A physico-mechanical approach to modeling of metal forming processes—Part II: damage analysis in processes with plastic flow of metals

Abstract A damage analysis is presented for the extrusion of a case-shaped cylindrical part by using a physico-mechanical approach for modeling metal forming processes. Two integral measures related to the hydrostatic and deviatoric parts of the damage tensor are used for the calculation of strain damage. The combined use of two damage measures in contrast to only one allows us to assess not only a risk of macro-fracture of the deformed material but also the stage of formation of large cavernous defects due to coalescence of ellipsoidal voids. Such a refined prediction of the actual quality of the material’s micro-structure is important when producing metalware that is supposed to operate under intense loading and thermal conditions. In case study of this paper the kinetic equations of damage are solved by using mutually consistent fields of stresses, flow velocities, and strains. It is shown that the predicted damage is less than its permissible value since a high hydrostatic pressure in the plastic zone heals the micro-defects, prevents their growth, and, thereby, increases the processing ductility of deformed metals during extrusion.

Keywords Plasticity · Plastic flow · Deformation · Plastic strain and stress · Slip line · Velocity field · Micro-structure · Strain induced damage · Meso-parameters · Damage equation · Metal forming · Extrusion · Case-shaped part

1 Introduction

The physico-mechanical approach to modeling Metal Forming (MF) processes as developed in Part I of this paper [21] is now applied to the analysis of a process with axisymmetric plastic flow. The constitutive equations describing large plastic deformations under complex loading (with changeable Lode angle) are based on both plastic flow theory as well as damage mechanics. The definition offered for the symmetric second-rank
order tensor of damage is physically motivated since its hydrostatic and deviatoric parts describe the evolution of damage connected with a change in volume and with the shape of the micro-defects, respectively. Consequently, this representation of damage kinetics requires us to use two integral measures \((\omega_1, \omega_2)\) for the calculation of damage in deformed materials: The measure \(\omega_1\) determines plastic dilatation related to an increase in void volume. It allows for a quantitative assessment of the risk of metal fracture caused by a critical amount of plastic dilatation \((\varepsilon_{cr}^\omega)\). In physical terms the measure \(\omega_1\) is identically equal to the measure \(\omega\) in dissipative damage theory [6] and is connected with the void volume fraction \((f_\omega)\) that is used as damage measure in some models [3,12,13]. The measure \(\omega_2\) accounts for the deviatoric strain of voids which is connected with a change in their shape \((\mathbf{e})\). The critical deformation of ellipsoidal voids corresponds to their intense coalescence and to the formation of large cavernous defects. It was demonstrated experimentally that the shape of the voids strongly affects their coalescence and, consequently, the formation of cavities in deformed material [4,8,11]. The critical deformation of ellipsoidal voids \((\varepsilon_{cr})\) corresponds to their intense coalescence and to the formation of large cavernous defects. Thus, we may say that the measure \(\omega_2\) assesses a risk of material microfracture at the mesolevel. The use of \(\omega_2\) is advantageous when producing metalware that is required to operate under intense loading and thermal actions, high pressures and strain rates. Such products and components are generally used in aerospace, automotive, and energy engineering.

Many complicated problems in research and development of MF techniques are still insufficiently investigated for products with irregular shape. In particular, for extrusion, where the processed material is under complex loading with strong variations of the stress state, it is very difficult to analyze and optimize the full process. The choice of an extrusion process for the case study of this paper is explained by its wide use in MF. The compressive regime of the stress state during extrusion promotes high ductility of the processed materials and, accordingly, a greater operational deformation. Therefore, extrusion (as a finishing operation) provides high-strength properties of products due to strain hardening. Specifically, we shall present in this paper a numerical analysis of the extrusion of a case-shaped part together with the prediction of its damage by using the aforementioned two damage characteristics, \(\omega_1\) and \(\omega_2\).

In the next chapter we will, first, briefly state the axisymmetric MF-boundary value problem and, second, outline how it can be solved numerically and, in particular, how the two damage measures can be determined. This will then be followed by a results section including a discussion of the outcome of the analysis: Chapter 3. We would like to point out that the extensive experience of the two Russian authors in the method of characteristics of flow lines was used to achieve this task. Clearly, in principle, it would have been possible using other numerical techniques instead, e.g., the Finite Element Method (FEM), maybe even in context with commercial FE-codes and user-defined subroutines. However, this has not been done in order to demonstrate the feasibility of the approach in due course. Alternative analyses are left to future research. Moreover note that we will frequently refer to Part I of this paper [21] where further details of the underlying damage theory are explained.

2 The constitutive equations for damage analysis

Extrusion is a manufacturing process in which a blank of material, usually a metal, is pressed in a forming container by the mechanical action of a punch [18]. The container, whose shape is the inverse of the shape of the product, forces the solid material to yield in the backward direction of the pressing punch (backward extrusion). Extrusion results in high strength of the processed material as a consequence of strain hardening. Furthermore, a major advantage of the extrusion process is a highly compressive stress state, i.e., stress triaxiality \((\bar{\sigma}) < -1\). Such a state of stress promotes a certain healing of the material, e.g., cracks and voids accumulated during MF may close, and, therefore, the obtainable degree of deformation is comparatively high [15]. Note that this is in contrast to other forming processes, such as deep drawing, where a sheet of metal is pressed into a forming die. In typical deep drawing processes stress triaxiality rises up to \((\bar{\sigma}) \approx +2/3\) [2]. Consequently, at approximately 60% of straining the accumulated micro-defects result in macroscopic damage. In contrast to that the compressive regime of backward extrusion allows for straining up to 90% and more.

In the following, we will analyze the state of stress and the material properties of an axisymmetric case-shaped part made of a cylindrical Al-Mg bar (cf., Fig. 1, right). Specifically, two integral measures of strain damage, \(\omega_1\) and \(\omega_2\), connected with the second-rank tensor of damage \(\omega_{ij}\), will be used for the meso-structural parameters of the plastic material during the solution of the case study in this paper. The kinetic equation for the damage measure \(\omega_1\), which is related to plastic dilatation of the deformed material due to an increase in
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Fig. 1 Some typical axisymmetric case-shaped parts obtained by extrusion (left); draft for calculations (right)

The kinetic equation for the damage measure $\omega_2$, which is connected with a change in micro-defect (void) shape under plastic deformation, takes on the following form:

$$\frac{d\omega_2}{dt} = \frac{\dot{\bar{e}}}{\bar{e}_{cr}} \equiv 1/2 \left( \frac{\dot{z}_i \cdot \dot{z}_j}{\bar{e}_{ij} \bar{e}_{ij}} \right),$$

where $\dot{\bar{e}}$ is the equivalent deviatoric strain rate of voids, $\bar{e}_{cr}$ is the critical deviatoric strain of voids corresponding to a stage of intense coalescence of ellipsoidal voids and formation of cavernous defects, and $\dot{z}_i$ are the mixed components of the deviatoric strain rate of voids.

The damage measures are defined over the following range: $\omega_1, \omega_2 \in [0; 1]$, where the upper-range value $\omega_1 = 1$ corresponds to the moment of macro-fracture, and $\omega_2 = 1$ corresponds to the stage of formation of cavernous defects (i.e., a stage of the macro-destruction at meso-scale). It should be noted that in [21] the authors related their own tensorial model of damage to commonly known models of void growth (e.g. [3,6,12,13]). The connection between the volume fraction of voids, $f_v$ (considered as a measure of damage in a number of known models), the criterion of damage, $\omega$ (a dissipative damage theory, e.g., [6]), and the authors’ measure of damage, $\omega_1$ (a tensorial theory), is discussed there. Two classical approaches (criteria $f_v$ and $\omega$) were compared with a tensorial theory (i.e., in terms of the measures $\omega_1$ and $\omega_2$).

As outlined in detail in [21] the evolution of the yield stress $\sigma_s$ can be described by

$$\sigma_s = \sigma_s^{(is)} \exp \left[ -\alpha \left( \frac{T - T_0}{T_{\text{max}} - T_0} \right)^q \right],$$

where $\sigma_s^{(is)} = \sigma_s^{(is)}(e_i, \dot{e}_{i0}, T_0, \mu_k)$ refers to isothermal hardening curves obtained for various materials at a fixed strain rate $\dot{e}_{i0}$ and an initial thermodynamic temperature $T_0$, $e_i$ is an intensity of the cumulative strains,
Therefore Eq. (1) becomes

\[ \varepsilon \sigma = \varepsilon_{\text{strain damage}} \text{related to plastic dilatation induced by micro-defect growth, and } \varepsilon_{\text{limit cumulative strain at shear, } \Lambda_{\text{limit}}} \text{ appears in Eq. (7), corresponds to the destruction of the deformed material. The limit strain } \Lambda_{\text{limit}} \text{ is determined by experimental diagrams of plasticity plotted for the investigated Al-Mg alloy under prescribed temperature-speed conditions (cf., Fig. 11 in [21]). Plasticity diagrams represent the experimentally determined dependence of the limit strain } \Lambda_{\text{limit}} \text{ on stress triaxiality } \langle \tilde{\sigma} \rangle, \text{ which can be approximated by a power function in the following form:} \]

\[ \Lambda_{\text{limit}} = A \cdot \exp(-c \langle \tilde{\sigma} \rangle) + B \cdot \exp(-c \langle \tilde{\sigma} \rangle), \]

where \( A, B, \text{ and } c \) are the experimentally determined parameters (cf., Table 2).

Proceeding in Eq. (7) with small finite increments of the strain \((\Delta \Lambda)\) we may find the material damage after the \( k \)th stage of deformation by numerical integration:

\[ \omega_{1k} = \omega_{10} + \sum_{j=1}^{k} \Delta \omega_{1j}, \quad \Delta \omega_{1j} = \frac{a \Lambda_{\text{limit}}^{a-1}}{\Lambda_{\text{limit}}^{a} \langle \tilde{\sigma} \rangle_{j}} \Delta \Lambda_{j}, \]
where \( \omega_{10} \) is the damage of the as-delivered material (cf., Table 2), \( \Lambda_j \) is the strain accumulated by the material particles up to the \( j \)th stage, \( \Lambda_{\text{lim}}(\bar{\sigma}_j) \) is the limit strain corresponding to stress triaxiality \( \bar{\sigma}_j \) at the \( j \)th stage.

Calculations of the damage measure \( \omega_2 \) by means of numerical integration of the kinetic equation (3) are connected with the determination of the equivalent deviatoric strain rate of voids \( \dot{h} \), and the critical deviatoric strain \( \dot{h}_c \), corresponding to a stage of intense coalescence of ellipsoidal voids and formation of cavernous defects. Calculations of the equivalent strain rate of voids necessitate an experimental determination of their dimensions changing under deformation [11,19]. The determination of the critical equivalent strain \( \dot{h}_c \) of voids in the investigated materials is based on microscopic analysis of void coalescence in test specimens during their stage-by-stage plastic deformation. The obtained micrographs will allow us to detect a stage of intense void coalescence into large cavernous defects. Such experiments involve great technical difficulties [10]. This is why we shall make use of statistical characteristics of void formation.

For instance, the averaged equivalent strain rate can be applied within each Representative Volume Element (RVE). A corresponding measure is the equivalent strain rate \( \dot{h} \) of the RVE. The hypothesis that it is possible to model the void deformation by using the strain measures of the RVE requires detailed experimental verification. The experimental justification of this hypothesis will allow us to predict shape changes and coalescence of voids by means of accompanying axes \( \xi^j \) plotted as coordinate grids on deformed specimens and manufacturing blanks. This hypothesis allows us to accept in our case \( \dot{h}(r, z, t) = \dot{h}(r, z, t) \), where the function \( \dot{h}(r, z, t) \) can be established by a distribution of the equivalent strain increments \( \Delta \dot{h}(r, z, h) \) in cells of the coordinate grid, i.e., \( \dot{h}(r, z, t) \equiv \Delta \dot{h}(r, z, h)/\Delta t = \dot{v}_h \cdot \Delta \dot{h}(r, z, h)/\Delta h \).

In order to find the critical equivalent strain of voids, \( \dot{h}_c \), the experimental results obtained by Bogatov et al. [1] were used. They investigated void coalescence and formation of cavernous defects in some structural metals at forming. Electron-probe analysis of the micro-structure of stepwise deformed metals allowed them to verify that the critical equivalent strain of voids can be expressed as \( \dot{h}_c = 0.6 \cdot e_{\text{lim}} \exp[0.05(1 - (\bar{\sigma}))] \), where \( e_{\text{lim}} = \Lambda_{\text{lim}}/2 \) is the limit deviatoric strain of RVE corresponding to the moment of its destruction. In view of the given experimental data Eq. (3) becomes

\[
d\omega_2 = \frac{5 \cdot \exp[0.05 ((\bar{\sigma}) - 1)]}{3 \cdot e_{\text{lim}}} d\dot{h}.
\]

(10)

By using small finite strain increments, \( \Delta \dot{h} \), in Eq. (10) we may calculate the material damage \( \omega_2 \) after the \( k \)th stage of deformation by numerical integration

\[
\omega_{2k} = \omega_{20} + \sum_{j=1}^{k} \Delta \omega_{2j}, \quad \Delta \omega_{2j} = \frac{5 \cdot \exp[0.05 ((\bar{\sigma})_j - 1)]}{3 \cdot e_{\text{lim} j}} (\Delta \dot{h})_j,
\]

(11)

where \( \omega_{20} \) is the damage of the as-delivered material (cf., Table 2), and \( e_{\text{lim} j} \) is the limit deviatoric strain corresponding to the stress state parameter \( (\bar{\sigma})_j \) at the \( j \)th stage of deformation.

The cumulative strain \( \Lambda \) (including the equivalent strain \( e = \Lambda/\sqrt{3} \)) and stress triaxiality \( \bar{\sigma} \) appears in the constitutive equations for damage measures (1), (7), (8), (10). In turn, the cumulative strain \( \Lambda \) and stress triaxiality \( \bar{\sigma} \) can be determined in each nodal point of the deformed material by using mutually consistent fields of stresses, \( \sigma_{ij} \), and plastic flow velocities, \( v_i \). Hence, damage analysis in processes with plastic flow of metals is a coupled problem of the calculation of stresses, flow velocities, and strains as well as mechanical and meso-structural characteristics of the deformed material.

To calculate the stress-strain state we use the equations describing axisymmetric plastic flow of metals in Eulerian cylindrical coordinates, where \( r, z, \theta \) denote the radial, the axial, and the circumferential direction, respectively. The basic equations of plasticity are given by the differential equations of equilibrium

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_0}{r} = 0, \quad \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \tau_{rz} = 0,
\]

(12)

the von Mises yield surface

\[
(\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\theta)^2 + (\sigma_\theta - \sigma_r)^2 + 6\tau_{rz}^2 = 6\tau_\theta^2,
\]

(13)
the condition of coaxiality of the strain rate deviator, $\dot{e}_{ij}$, and the stress deviator, $s_{ij}$:

$$\frac{\partial \nu^r}{\partial z} + \frac{\partial \nu_z}{\partial r} = \frac{\partial \nu^r}{\partial z} - \frac{\partial \nu_z}{\partial r},$$

(14)

the condition of similarity of the deviators $\dot{e}_{ij}$ and $s_{ij}$ (15), i.e., of coincidence of their Lode angles, $\phi_\theta$ and $\phi_\sigma$ [21]

$$\sqrt{\frac{I_2(\dot{e}_{ij})}{I_2(s_{ij})}} = \sqrt{\frac{I_3(\dot{e}_{ij})}{I_3(s_{ij})}},$$

(15)

and the incompressibility condition

$$\frac{\partial \nu^r}{\partial r} + \frac{\partial \nu_z}{\partial z} + \nu_r = 0,$$

(16)

where $\sigma_r$, $\sigma_z$, $\sigma_\theta$, $\tau_{rz}$ are nonzero components of the stress tensor $\sigma_{ij}$, $\tau_\delta$ is the yield stress for shear, $\nu_r$, $\nu_z$ are the components of the vector of plastic flow velocity, $I_2(\dot{e}_{ij}) = \dot{e}^2$, $I_3(s_{ij}) = s^2$, $I_3(s_{ij})$ are the second and the third invariants of deviatoric strain rates $\dot{e}_{ij}$ and deviatoric stresses $s_{ij}$, respectively, and $\dot{e}$, $s$ denote the equivalent deviatoric strain rate and stress.

The deformed material is considered as a rigidly-plastic solid because plastic strains amount to 70–90% at MF being $\approx 10^3$ larger than elastic strains [21]. The axisymmetric stress-strain state has the following features: for the stress components we have $\tau_{z\theta} = \tau_{theta} = 0$, $\nu_\theta = 0$. The hoop stress $\sigma_\theta$ is a principal stress. In order to represent the axisymmetric stress-strain state completely it will be sufficient to determine a field of stresses and flow velocities in one of the meridian cross-sections of the deformed solid.

For the analysis of MF processes with a rapidly changing stress state the basic differential equations for axisymmetric plastic flow are solved by hyperbolic approximations [16]. The technique is based on a representation of yield zones in a special form for the stresses related to Iljushin’s deviatoric stress space [5]. Actually, this technique represents the enhanced slip line method for solving the problems of axisymmetric plastic flow with the intense change of stress state. The information on slip line fields (i.e., directions of microscopic shear bands) in plastic zone of the deformed material proves to be very useful when modeling the nucleation of voids. It is experimentally known that large voids generate microscopic shear bands. Voids nucleate, grow, and coalesce within these shear bands, finally forming large cavities (20–30 μm) [20].

The slip lines form two families of mutually orthogonal lines ($\alpha$ and $\beta$) in the meridian cross-section of the plastic zone

$$\frac{dz}{dr} = \tan \delta_\theta \text{ (at line } \alpha), \quad \frac{dz}{dr} = -\tan \delta_\theta \text{ (at line } \beta),$$

(17)

where $\delta_\theta$ is the angle between the $r$-axis and the line $\alpha$. The angle $\delta_\theta$ is connected with the stress components by the following relation:

$$\tan 2\delta_\theta = -\frac{\sigma_r - \sigma_z}{2\tau_{rz}},$$

(18)

which can be used in the basic solution as the additional condition in terms of stresses.

When solving the basic equations (12)–(16) it is convenient to use a special representing space of the parameters $\phi_\theta$, $m_\theta$, $m_r$, $m_z$ which are introduced by the following relations:

$$\tan 2\phi_\theta = \sqrt{\frac{2}{3}} \tan 2\delta_\theta, \quad s m_\theta \cos 2\phi_\theta = \tau_{z\theta} \cos 2\delta_\theta$$

$$I_1(\tilde{s}_{ij}) = m_\theta \sin 2\phi_\theta - m_z \sin (\sigma_z - \sigma_\theta), \quad I_2(\tilde{s}_{ij}) = m_r \sin (\sigma_\theta - \sigma_r), \quad I_3(\tilde{s}_{ij}) = m_r \sin (\sigma_\theta - \sigma_r)$$

(19)

where $I_1(\tilde{s}_{ij}) = 0$, $I_2(\tilde{s}_{ij}) = 1$ are the first and the second invariants of the directing stress deviator $\tilde{s}_{ij} = s_{ij}/\sqrt{I_2(s_{ij})}$, and sign$(\sigma_z - \sigma_\theta)$, sign$(\sigma_\theta - \sigma_r)$ are the sign functions, i.e., the signs of the arguments $(\sigma_z - \sigma_\theta)$ and $(\sigma_\theta - \sigma_r)$.

The representing space of the parameters $\phi_\theta$, $m_\theta$, $m_r$, $m_z$ is connected with Iljushin’s deviatoric space [5]. The sign functions sign$(\sigma_z - \sigma_\theta)$ and sign$(\sigma_\theta - \sigma_r)$ that appear in Eqs. (19) depend on the type of processing operations. In processes with predominant axial compression (e.g., extrusion, upsetting, bulk forging) we have $\sigma_r \geq \sigma_\theta \geq \sigma_z$, sign$(\sigma_z - \sigma_\theta) = -1$, sign$(\sigma_\theta - \sigma_r) = -1$. 

\[\]
Physically speaking, the parameters \( m_\theta, m_r, m_z \) define a direction of the vector of the octahedral shear stress, \( \tau_8 \), which is within the deviatoric plane (19)\(^3\) [7]. Eqs. (12) in combination with the parametric representation of stresses (19) takes the following form in the system of coordinates \( \alpha \) and \( \beta \):

\[
\frac{\partial \langle \sigma \rangle}{\partial s_\alpha} - 2\tau_s m_\theta \left[ 1 + \sqrt{\frac{2}{3}} \psi'(\delta) \right] \frac{\partial \delta_\theta}{\partial s_\alpha} - \sqrt{\frac{1 - m_\theta^2}{3}} \text{sign}(\sigma_\theta - \sigma_\alpha) \frac{\partial \tau_s}{\partial s_\alpha} + m_\theta \frac{\partial \tau_s}{\partial s_\beta} + \sqrt{\frac{r}{3}} \sqrt{1 - m_\theta^2} \cos \delta_\theta \text{sign}(\sigma_\theta - \sigma_\alpha) \\
+ \tau_s \frac{m_\theta}{\sqrt{3}} \sqrt{1 - m_\theta^2} \sin \delta_\theta = 0,
\]

\[ (20) \]

\[
\frac{\partial \langle \sigma \rangle}{\partial s_\beta} + 2\tau_s m_\theta \left[ 1 + \sqrt{\frac{2}{3}} \psi'(\delta) \right] \frac{\partial \delta_\theta}{\partial s_\beta} + \sqrt{\frac{1 - m_\theta^2}{3}} \text{sign}(\sigma_\theta - \sigma_\alpha) \frac{\partial \tau_s}{\partial s_\beta} + \sqrt{\frac{r}{3}} \sqrt{1 - m_\theta^2} \sin \delta_\theta \text{sign}(\sigma_\theta - \sigma_\alpha) \frac{\partial m_\theta}{\partial s_\beta} \\
+ \tau_s \frac{m_\theta}{\sqrt{3}} \cos \delta_\theta = 0,
\]

\[ (21) \]

where \( \langle \sigma \rangle \) indicates hydrostatic stress and \( \psi'(\delta) = \frac{d\psi_\theta}{d\delta_\theta} \), \( m_\alpha = m_\beta = 1 - \frac{1}{\sqrt{2}} \sqrt{1 - m_\theta^2} \).

Equations (14) and (16) for the velocities can be rewritten in terms of fixed coordinates \( \alpha^* \) and \( \beta^* \) coinciding with trajectories \( \alpha \) and \( \beta \):

\[
\frac{\partial \nu_\alpha^*}{\partial s_\alpha} - \frac{\nu_\alpha^* \cos \delta_\theta - \nu_\beta^* \sin \delta_\theta}{r} = 0,
\]

\[ (22) \]

\[
\frac{\partial \nu_\beta^*}{\partial s_\beta} - \frac{\nu_\alpha^* \cos \delta_\theta - \nu_\beta^* \sin \delta_\theta}{2r} = 0.
\]

\[ (23) \]

The solution of Eqs. (15), (20)–(23) satisfy Cauchy-type boundary conditions. In case of an axisymmetric problem the Cauchy-type conditions are given by boundary values of the required functions \( \langle \sigma \rangle, \delta_\theta, m_\alpha, \nu_\alpha^*, \nu_\beta^* \) at the two-dimensional surface \( \omega(\alpha^*, \beta^*) = 0 \) in the space \( \alpha^*, \beta^* \).

The four partial differential equations (20)–(23) are closed w.r.t. five required functions \( \langle \sigma \rangle, \delta_\theta, m_\alpha, \nu_\alpha^*, \nu_\beta^* \) and \( \nu_\beta^* \) by means of the differential constraint (15). These four equations are of hyperbolic type and have two pair-wise coincident sets of mutually orthogonal characteristics, \( \alpha \) and \( \beta \). The characteristic lines \( \alpha \) and \( \beta \) coincide with the slip lines and they are described by Eq. (17). This allows us to use grid-characteristic schemes for the numerical solution of arising boundary problems.

3 Results and discussion

Since the characteristics of damage depend on strain history a stepwise analysis of the forming process together with a detailed determination of stress-strain state is necessary for their calculation. For the purpose of the analysis the process of extrusion will be divided into three stages. The first stage is characterized by contraction of the blank with the initial height, \( h_0 = 33.1 \) mm, and filling of the space between the container and the blank. It ends with outflow into the clearance. During this stage the height of the blank slightly decreases with a typical degree of deformation of \( \epsilon_1 = 0.01 - 0.02 \). Here we put \( \epsilon_1 = 0.018 \) and thus we obtain for the difference of initial and final height in the first stage, \( \Delta h_1 = \epsilon_1 \cdot h_0 = \Delta h_1 = 0.60 \) mm. The second stage is characterized by a constant shape and size of the plastic zone. Plastic flow at this stage is stationary with a decreasing height of the blank and an outflow into the clearance between the container and the punch. In the third stage the plastic zone spreads over all the material under the punch. The plastic flow is now non-stationary.
The stress-strain state is evaluated at four stages (cf., Fig. 2), as follows:

1. at the initial moment of the second stage when $\Delta h_1 = 0.60$ mm,
2. at an intermediate moment when $\Delta h_2 = 7.45$ mm,
3. at an intermediate moment when $\Delta h_3 = 14.30$ mm,
4. at the final state, i.e., $\Delta h_4 = 21.10$ mm.

By solving the boundary problem we may construct a field of slip lines (i.e., characteristics) in the meridian cross-section of the axisymmetrically deformed material. Then the contour of the plastic zone is determined, and stress components are calculated (cf., Fig. 3).

Note, that the defining differential equations (2) and (20)–(23) are represented in recursive form for numerical calculations [17]. When solving these boundary value problems we use a condition of maximum contact friction at the end surface of the punch and at the back surface of the container, i.e., $\tau_c = \tau_{\alpha\beta}$ ($\tau_c$ denotes the shearing stress at the contact). The normal pressure of the deformed metal on a lateral surface of the container is much lower than on its back surface. Therefore, a condition of the Prandtl-type for non-maximum contact friction is realized at a lateral surface of the container

$$\tau_c = f_p \tau_{\alpha\beta} = f_p m_\theta \tau_s,$$  \hspace{1cm} (24)

where $f_p \in [0; 1]$ is a coefficient of plastic friction.

Then the flow velocity fields are determined at the observed moments of extrusion, (cf., Fig. 4). The known field of slip lines ($\alpha$, $\beta$) allows us to use the velocity equations (22) and (23) when determining the velocities $v_\alpha^*$, $v_\beta^*$ in the nodal points, since the angle $\delta_\theta$ and the coordinate arcs $s_\alpha$, $s_\beta$ are already known from the solution for the stress components. The velocity fields in Fig. 4 are determined by solving the boundary problems. A grid of lines $\bar{\alpha}$, $\bar{\beta}$ selected in the plastic zone consists of continuum points. This grid is coincident with a grid of the trajectories (or characteristics) $\alpha$, $\beta$ at the observed moment of deformation. This grid of the material lines $\bar{\alpha}$, $\bar{\beta}$ can be considered as an associated coordinate frame [14], and the constructed velocity field (cf., Fig. 4a) can be considered as a representation of the material lines $\bar{\alpha}$, $\bar{\beta}$ in the velocity plane. A similar representation of the flow velocity field is essential when modeling the non-stationary processes of plastic deformation. The kinematic relations (20) and (21) can be applied to the continuum lines $\bar{\alpha}$, $\bar{\beta}$ which are coincident with the slip lines $\alpha$, $\beta$. During non-stationary flow a grid of the slip lines $\alpha$, $\beta$ [characteristics of Eqs. (20)–(23)] is displaced w.r.t. the material lines $\bar{\alpha}$, $\bar{\beta}$.
Fig. 3 Extrusion of the axisymmetric case-shaped part (final non-stationary stage): the plastic zone, stress field, flow lines for various punch displacements: a $\Delta h_1 = 0.60$ mm; b $\Delta h_2 = 7.45$ mm; c $\Delta h_3 = 14.30$ mm; d $\Delta h_4 = 21.10$ mm

Fig. 4 The velocity field during extrusion of the axisymmetric case-shaped part (the final non-stationary stage) for various punch displacements: a $\Delta h_1 = 0.60$ mm; b $\Delta h_2 = 7.45$ mm; c $\Delta h_3 = 14.30$ mm; d $\Delta h_4 = 21.10$ mm
The velocity field has the following typical feature for an axisymmetric process: The angle between the lines \( \bar{\alpha} \) and \( \bar{\beta} \) increasingly changes from \( \pi/2 \) to \( \pi \), when we map the lines \( \bar{\alpha}, \bar{\beta} \), which are close to the symmetry axis of the blank, onto the plane of velocities \( \nu_r, \nu_z \). Thus the representation of the material lines \( \bar{\alpha}, \bar{\beta} \) in the plane of velocities \( \nu_r, \nu_z \) depends on the Lode angle.

The degree of conformity between the fields of stresses \( \sigma_{ij}^{(0)} \) and velocities \( \nu_i^{(0)} \) in the basic solution is determined by fulfilling the similarity condition (15) for the deviators \( \dot{e}_{ij} \) and \( s_{ij} \) in the form

\[
m_\theta^{(\dot{\varepsilon})} - m_\theta^{(\sigma)} \leq [\Delta m_\theta] ,
\]

[\( \Delta m_\theta \)] being a permissible mismatch error for the parameters \( m_\theta^{(\dot{\varepsilon})} \) and \( m_\theta^{(\sigma)} \) related to the fields of stresses and strain rates.

For the solution of the inequality (25) the method of group relaxation is used (e.g., [9, Section 20.3–2]). By regulating the absolute value of the difference between the parameters \( m_\theta^{(\dot{\varepsilon})} \) and \( m_\theta^{(\sigma)} \) in selected nodes of the plastic zone (instead of bringing it down to zero in the first step) it is possible to fulfill the inequality (25) already at the first correction with the permissible error \( [\Delta m_\theta] = 0.01 \). The calculated stress field (cf., Fig. 3) and the corresponding field of flow velocities (cf., Fig. 4) satisfy the inequality (25) for the permissible error \( [\Delta m_\theta] = 0.01 \).

The fields of stresses \( (\sigma_{ij}) \) and the plastic flow velocities \( (\nu_i) \) found at the different stages of non-stationary deformation allow us to calculate cumulative strains \( (\Lambda) \). Cumulative strains \( \Lambda \) appear in the kinetic equations for meso-structural parameters \( (\mu_k) \) of the deformed material, including the damage parameters \( \omega_1 \) and \( \omega_2 \). Cumulative strains are calculated along trajectories of movement of the material particles. The cumulative strains after stage \( k \) are \( \Lambda_k = \sum_{j=1}^{k} \Delta \Lambda_j \), and \( \Lambda = \sum_{j=1}^{n} \Delta \Lambda_j \) at the end of forming. In Fig. 6 the distribution of the cumulative strains \( \Lambda \) is shown for the radially directed middle layer of bottom part of the finished product \((z = 0.5h_b)\), where \( h_b \) is bottom thickness.

In order to verify the calculated strain distribution the strains are experimentally determined by means of a coordinate grid. This technique allows us to find fields of the shear strains \( \Lambda(r, z) \) at different stages of extrusion. The fields \( \Lambda(r, z) \) are used when determining the yield stress \( \sigma_{y}^{(0)}(r, z) \) from the intensity of the cumulative strains \( \varepsilon_{\tau}(r, z) = \Lambda(r, z) / \sqrt{3} \), cf., Eq. (5), and, moreover, for finding the yield stress under shear \( \tau_{s,n}(r_{m,n}, z_{m,n}) = \sigma_s(r_{m,n}, z_{m,n}) / \sqrt{3} \). The specimen is cut through its meridian cross-section in order to plot a coordinate grid (cf., Fig. 5). In Fig. 6 (left) the experimentally found distribution of the cumulative strains \( \Lambda \) is shown for the radially directed middle layer of bottom part of the finished product \((z = 0.5h_b)\). The calculated distribution of \( \Lambda \) satisfactorily fits the experimental results.

Now we have the complete information which is required to calculate damage measures using Eqs. (9) and (11). Moreover, we know the mechanical and meso-structural parameters of the Al-alloy (cf., Tables 1 and 2) as well as stepwise values of the strain, \( \Delta \Lambda_j \), and the limit strain \( \Lambda_{\text{lim}} \) (as the known function of stress triaxiality \( (\bar{\sigma}) \), cf., (8)).

In Fig. 6 the predicted distribution of damage \( (\omega_1 \text{ and } \omega_2) \) is shown for the radially directed middle layer of bottom of the finished product (with the coordinate \( z = 0.5h_b \)). The obvious fact that the damage increases from the symmetry axis towards the walls can be explained as follows. First, the strain \( (\Lambda) \) accumulated by the material particles along their motion trajectories directed towards the walls increases in the same direction.

Second, large compressive hydrostatic stress \( (\bar{\sigma}) \) prevents the evolution of damage, and indeed, the absolute value of \( (\bar{\sigma}) \) increases when moving from the walls toward the axis of symmetry.

Also note in Fig. 6 that the growth rate of \( \omega_2 \) is higher than that of \( \omega_1 \). The ratio of the increments \( d\omega_2/d\omega_1 = 1 \) follows from Eqs. (7) and (10). For example, \( d\omega_2/d\omega_1 = 1.3 - 1.6 \) when the triaxiality is \( (\bar{\sigma}) = -3 - 0.67 \), which is typical for MF processes. In other words, this ratio is greater for tensile regimes of the stress state \( (\bar{\sigma}) > 0 \) than for compressive ones \( (\bar{\sigma}) < 0 \). These relations are physically obvious. Void coalescence always advances macro-crack formation. And during tensile loading regimes (when tensile stresses promote void growth) this advance will be greater than during compressive ones.

The predicted values of damage induced by micro-defects are much smaller than the allowed ones: \( \omega_{1\text{lim}} = 0.42, \omega_{2\text{lim}} = 1 \). High hydrostatic pressure in the plastic zone heals micro-defects prevents their growth and thereby enhances the processing ductility of the deformed metal. Thus, larger processing strains are possible during extrusion (up to 90% and more) when compared to deep-drawing (no more than 70–75%) when producing axisymmetric case-shaped parts.
Fig. 5 Coordinate grid in a meridian cross-section of the case-shaped Al-Mg part at the final stage of extrusion

Fig. 6 Distribution of the strain $\Lambda$ (left) and the damage measures $\omega_1$, $\omega_2$ (right) in the median layer ($z = 0.5h_b = 6\text{mm}$) of the meridian cross-section of the axisymmetric case-shaped Al-Mg part (with the outer radius $r_c = 57\text{ mm}$) after extrusion

4 Conclusions and outlook

In this section we will discuss some results of the analysis of extrusion of the axisymmetric case-shaped part. The type of the state of stress strongly varies within the volume of the processed metal from monaxial compression at its symmetry axis to simple shear around the clearance between the punch and the container. This fact indicates complex loading of the deformed material, i.e., a rotation of the principal axes of the stress tensor w.r.t. the deforming material lines and a change of proportions between the components of the stress tensor. Conditions of complex loading necessitate using the associated rule of plastic flow for the basic equations.

Cumulative strains are irregularly distributed in the bottom part of the finished product. The largest strain is accumulated by the material particles which appear out of the plastic zone or approach to its neighborhood at the final moment of processing. The reliability of the predicted mechanical and mesostructural characteristics depends on the accuracy of the calculated strain field. The strains that were experimentally determined from a coordinate grid satisfactorily agree with the theoretically calculated strains.
Comparison between the maximum values of damage measures and their allowed values, i.e., $\omega_{1\text{max}} = 0.42 < \omega_{1\text{lim}} = 1$, $\omega_{2\text{max}} = 0.51 < \omega_{2\text{lim}} = 1$, allow us to predict the quality of the material structure after extrusion. Large processing strain at extrusion provides high-strength properties of products due to strain hardening.

The combined use of two damage measures, $\omega_1$ and $\omega_2$, in contrast to using only $\omega_1$ or the volume fraction of voids, $f_v$, as in the known common models (cf., e.g., [3,6,12,13]), allows us to predict not only a risk of macro-fracture of the deformed material but even the stage of formation of large cavernous defects due to coalescence of voids taking a change in their shape and orientation into account. For example, when applying deep-drawing operations (with triaxiality $\langle \tilde{\sigma} \rangle > 0$) for manufacturing of axisymmetric case-shaped parts under large processing deformations it can happen that $\omega_1 < 1$ while $\omega_2 = 1$. In this case using only one measure ($\omega_1$ or $f_v$) for calculations would indicate non-criticality of damage while the second measure ($\omega_2 = 1$) reveals a critical stage of voids coalescence and generation of cavities. This situation is undesirable or even unacceptable when producing metalware to be operated under intense loading and thermal actions which is widespread in aerospace, automotive, and energy engineering.

It should be noted once again that a successful practical application of the tensor theory to modeling of MF processes requires rather laborious experimental research on damage kinetics for deformed materials under complex loading. Such experiments will promote the creation of a database for meso-structural properties of plastically deformed materials which is necessary for computer simulations. To this end the authors have already begun their experimental research on the growth, shape change, and coalescence of meso-defects during stepwise plastic deformation of some structural metals. Tensile tests of sheet Al specimens with mock voids (pre-machined micro-holes) gave the following results: Voids take on a precise ellipsoidal shape elongated in the line of the principal strain axis (cf., Fig. 7). The strain $\tilde{\varepsilon}_{\text{void}}$ of ellipsoidal voids is appreciably larger than the strain $\varepsilon_{\text{RVE}}$ of the material element (RVE) containing these imitation defects: $\tilde{\varepsilon}_{\text{void}}/\varepsilon_{\text{RVE}} \approx 1.4 - 1.5$. It could be verified that large imitation voids are the sources of localized strain bands which propagate in the direction of 55–60° to the axis of elongation. The commercial FE code ABAQUS was applied for processing the obtained experimental data on damage of strained materials. An example of FE calculations of equivalent plastic strains during uniaxial tension of the sheet Al specimen with imitation meso-defects is presented in Fig. 7. Such experimental data obtained for various types of the stress-strain state will allow us to realize a comparative assessment of different models of ductile damage in the near future.

In a single paper it is obviously impossible to discuss (even in a compressed form) all the questions pertinent to the presented physico-mechanical approach. The enhanced slip line method was used here in order to obtain the rapid and accurate solution of the applied problem. In future work the authors aim at coupling the constitutive equations of their tensorial theory with FE codes for MF processes.
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