3.2.4 Eulerian Equations, Angular Momentum Theorem in a Rotating Coordinate Frame

As we saw, the rotational motion of a rigid body can be described by the angular momentum theorem. The derivative of the angular momentum with respect to time had to be calculated in a fixed coordinate system, the inertial system.

The extremely big disadvantage of this procedure is that if a body rotates but the coordinate system is fixed you always have to calculate the mass moments of inertia and the products of inertia with respect to this fixed coordinate system but while the body is rotating you may have different moments and products of inertia at every time instant. This makes the practical use of the angular momentum theorem very difficult.

The way out of this problem is to find a moving coordinate system (e.g. a body-fixed coordinate frame) in which the moments and products of inertia remain constant during rotation.

The angular momentum theorem says that:

\[ \dot{L} = M \]

To indicate which coordinate frame is used, we introduce the index \( I \):

\[ I \dot{L} = I M \]

The components in the \((I)\)-frame can be transformed to the rotating frame \((R)\) using the rotation matrix \( A_{IR} \) (as we did many times before):

\[ \frac{d}{dt}(A_{IR} \kappa L) = (A_{IR} R M) \]

or

\[ A_{IR} (R \dot{L}) + (A_{IR})_R L = A_{IR} R M \]

Premultiplying by \( A_{KI} \) from the left side:

\[ (R \dot{L}) + A_{RI} (A_{IR})_R L = R M \]

In the second expression we recognize the matrix of the angular speeds (compare to chap. 2, kinematics of the rigid body):

\[ (R \dot{L}) + (R \tilde{\omega}_{RI} R L) = R M \]  \hspace{1cm} (3.2.34)

The angular speed are the rotational speeds of the rotating frame in components of the \(R\)-frame. The product of the angular velocities and the angular momentum can be written in classical formulation as a cross product.
\[
\frac{d'}{dt}(\mathbf{R}L) + (\mathbf{R}_R \omega \times \mathbf{R}L) = \mathbf{R}M
\]  \hspace{1cm} (3.2.35)

The first term \( \frac{d'}{dt}(\mathbf{R}L) \) is marked by the prime and indicates that the derivative is the temporal change in the rotating frame.

### 3.2.4.1 Coordinate System in the Center of Gravity

As we have seen, the expression for the angular momentum will be much simpler if the reference point is the center of gravity \( S \). In this special case we got:

\[
\mathbf{L}_S = \mathbf{J}_S \mathbf{\omega}
\]

Because the moments and products of inertia are constant with respect to a body-fixed rotating coordinate system, the derivative of \( \mathbf{L} \) with respect to time yields:

\[
(\mathbf{R}\mathbf{L}_S)' = \mathbf{J}_S \mathbf{\dot{\omega}}
\]

and putting this into eqn.(3.2.34) leads to

\[
\mathbf{J}_S \mathbf{\ddot{\omega}} + \mathbf{\omega} \times \mathbf{J}_S \mathbf{\dot{\omega}} = \mathbf{R}M_S
\]  \hspace{1cm} (3.2.36)

or shortly, leaving the R-subscript away:

\[
\mathbf{J}_S \mathbf{\ddot{\omega}} + \mathbf{\omega} \times \mathbf{J}_S \mathbf{\dot{\omega}} = \mathbf{M}_S
\]  \hspace{1cm} (3.2.37)

In classical notation this equation writes as

\[
\mathbf{J}_S \frac{d'}{dt} \mathbf{\omega} + \mathbf{\omega} \times \mathbf{J}_S \mathbf{\omega} = \mathbf{M}_S
\]  \hspace{1cm} (3.2.38)

### 3.2.4.2 Body-fixed Principal Axes

In body-fixed principal axes these equation become very simple in structure. The inertia matrix then becomes diagonal because the products of inertia are zero:

\[
\mathbf{J}_S = \begin{bmatrix}
J_1 & 0 & 0 \\
0 & J_2 & 0 \\
0 & 0 & J_3
\end{bmatrix}
\]
It follows that

\[
\begin{align*}
J_1\dot{\omega}_1 - (J_2 - J_3) \omega_2 \omega_3 &= M_1 \\
J_2\dot{\omega}_2 - (J_3 - J_1) \omega_3 \omega_1 &= M_2 \\
J_3\dot{\omega}_3 - (J_1 - J_2) \omega_1 \omega_2 &= M_3
\end{align*}
\]  

\text{(3.2.39)}

These are the so-called Eulerian equations for gyroscopic systems (Euler, 1765). They are three non-linear coupled differential equations. All components of \( J, \omega \) and moment \( M \) have to be written in components of the rotating (R)-coordinate frame. These equations have a great relevance for the derivation of the equations of motion of mechanical systems with rotating components.

3.2.6 Angular Momentum Theorem in a Guided Coordinate System

As already mentioned it can be more convenient to use neither a body fixed nor an inertial system to describe the kinematics of a rotating body.

The approach can be clarified with an example. A cylindrical roll is rotating about its rotation axis, which corresponds to the x-axis (Fig.3.9). The whole system is rotating again about the vertical z-axis. The guided coordinate system has its origin on the z-axis. The y-axis is always horizontal and does not –in contrast to the body fixed x'-y'-z'–system- rotate with the cylindrical roll, whereas x- and x'- axis are identical. The coordinate system in the Center of Gravity is -in contrast to the former chapter- not used here.

Fig 3.9: System with rotating cylindrical roll
The angular momentum theorem expressed in the guided coordinate system using the point of origin of the $R$-system as centre of reference is

$$\frac{d}{dt}(r L_\theta) + r \omega_{GCS} \times r L_\theta = r M_\theta$$  \hspace{1cm} (3.2.40)

whereas the first term is analogue to eqn. (3.2.34, 3.2.35) and describes the relative change of the angular momentum in the guided coordinate system (GCS). The second term of the left hand side is the change of angular momentum due to the rotation of the coordinate system. The expression on the right hand side is the moment expressed in coordinates of the guided $R$-system.

In this example the rotation of the guided coordinate system about the vertical rotation axis is expressed by following angular velocity vector

$$r \omega_{GCS} = \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix}$$  \hspace{1cm} (3.2.41)

The mass moments of inertia are:

$$J = \begin{bmatrix} J_{xx} & 0 & 0 \\ 0 & J_{yy} & 0 \\ 0 & 0 & J_{zz} \end{bmatrix}$$  \hspace{1cm} (3.2.42)

Due to the symmetry of the cylindrical rolls $J_{zz}$ is equal to $J_{yy}$, the off-diagonal terms are zero. The mass moment of inertia of a homogeneous cylinder (mass $m$, radius $r$, length $l$) is:

$$J_{xx} = \frac{1}{2} mr^2$$  \hspace{1cm} (3.2.43a)

$$J_{zz} = J_{yy} = \frac{1}{12} m \left( l^2 + 3r^2 \right) + ma^2$$  \hspace{1cm} (3.2.43b)

The last term in eqn. (3.2.43b) is the parallel axis part. The angular velocity vector of the rotor consists of two components: rotation about x-axis and rotation about z-axis:

$$R \omega = \begin{bmatrix} \omega_x \\ 0 \\ \omega_z \end{bmatrix}$$  \hspace{1cm} (3.2.44)

The angular momentum is

$$r L_\theta = J \ R \omega = \begin{bmatrix} J_{xx} \omega_x \\ 0 \\ J_{zz} \omega_z \end{bmatrix}$$  \hspace{1cm} (3.2.45)
With constant angular velocity in terms of the guided coordinate system

\[
\frac{d}{dt}(R L_0) = 0
\]

the moment can be determined to be

\[
R M_{\theta,0} = \begin{pmatrix}
0 \\
J_{xx} \omega_z \omega_z \\
0
\end{pmatrix} = - R M_{\theta,\text{gyroscopic}}
\]

(3.2.46)

The gyroscopic moment on the right hand side is the action of the rotor to the environment. It loads the bearings as well as other machine parts. In the present example the gyroscopic moment has only a y-component. Compared to the guided coordinate system the gyroscopic moment has negative sign, assuming that the two values of the angular velocities are positive.
3.3 Kinetic Energy of a Rigid Body

The kinetic energy of a particle is

\[ E_{\text{kin}} = \frac{1}{2} m v^2 \]  

(3.3.1)

For a rigid body the kinetic energy is obtained by integration over all infinitesimal element of mass \( dm \):

\[ E_{\text{kin}} = \frac{1}{2} \int v^2 \, dm \]  

(3.3.2)

Using the Eulerian kinematic equation (see Chapter. 2.2)

\[ \dot{v} = v_P = v_A + \bar{\omega} \times \bar{r}_{AP} \]

where \( A \) is an arbitrary reference point. We express the square of the velocities by:

\[ v^2 = v^T v = (v_A + \bar{\omega} \times \bar{r}_{AP})^T (v_A + \bar{\omega} \times \bar{r}_{AP}) \]

(3.3.3)

and put it into the integral. Further manipulations lead to

\[ E_{\text{kin}} = \frac{1}{2} m v_A^2 + m v_A^T (\bar{\omega} \times \bar{r}_{AS}) + \frac{1}{2} \bar{\omega}^T J_A \bar{\omega} \]

(3.3.4)

\( \bar{r}_{AS} \) is the position vector pointing from point \( A \) to the CG \( S \). This quantity results from the integral

\[ \int (\bar{r}_{AP} \, dm = \bar{r}_{AS} \, m \]

and is the static moment which defines the position of the CG.

In vector notation the kinetic energy is

\[ E_{\text{kin}} = \frac{1}{2} m v_A^2 + m v_A^T (\bar{\omega} \times \bar{r}_{AS}) + \frac{1}{2} \bar{\omega}^T J_A \bar{\omega} \]

We see that kinetic energy has three terms:

- The first is a pure translational part resulting from the velocity of point \( A \),
- The third describes the pure rotation related to point \( A \),
- And the middle term is a coupling term of translation and rotation. Don’t forget this part of the kinetic energy.
**Important Special Cases:**

1) If the reference point is the center of gravity: \( A = S \)

\( \mathcal{L}_{AS} = \mathcal{L}_{SS} = 0 \) (the coupling term vanishes) and we get:

\[
E_{\text{kin}} = \frac{1}{2} \mathbf{v}^2_S + \frac{1}{2} \omega^T \mathbf{J}_S \omega
\]

\( E_{\text{kin}} = E_{\text{trans}} + E_{\text{rot}} \)

In cartesian coordinates we get

\[
E_{\text{trans}} = \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2)
\]

(3.3.5a)

and

\[
E_{\text{rot}} = \begin{bmatrix}
J_{xx} & J_{xy} & J_{xz} \\
J_{yx} & J_{yy} & J_{yz} \\
J_{zx} & J_{zy} & J_{zz}
\end{bmatrix}
\begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix}
\]

\[
E_{\text{rot}} = \frac{1}{2} \left[ J_{xx} \omega_x^2 + 2J_{xy} \omega_y \omega_x + 2J_{xz} \omega_z \omega_x + 2J_{yx} \omega_y \omega_x + J_{yy} \omega_y^2 + J_{zz} \omega_z^2 \right]
\]

(3.3.5b)

For principal axes the rotational part of the energy is

\[
E_{\text{rot}} = \frac{1}{2} (J_1 \omega_1^2 + J_2 \omega_2^2 + J_3 \omega_3^2)
\]

(3.3.6)
2) Rotation about a fix-point $A$ : $\nu_A = 0$

\[ E_{\text{kin}} = \frac{1}{2} \omega^T J_A \omega \]  

(3.3.7)

pure rotation about point $A$.

3) Rotation about a fixed axis through $A$

The vector of the angular velocity is identical with the spinning axis. The whole expression reduced to one term

\[ E_{\text{kin}} = \frac{1}{2} J_A \omega^2 \]  

(3.3.8)

where $J_A$ is the moment of inertia with respect to the axis through point $A$.

### 3.4 Lagrange’s Equations of Motion

The Lagrangian equations of motion or the Lagrangian equations of the 2. kind (LE2)\(^1\) play a major role in dynamics. They belong to the class of analytical methods which analyze the equations of motions from a global kinetic energy and potential energy consideration. Usually, reaction forces do not appear. They are a standard tool to obtain the equations of motion of dynamic system. The LE2 can be derived from D’Alembert’s principle.

Our system consists of $n$ rigid bodies and has $f$ dofs. We need the relation between the position vector $r_i$ and the $f$ generalized coordinates $q_i$. As shown in chapter 2.1 we can write

\[ r_i = r_i(q_1, q_2, \ldots, q_f) \quad i = 1, \ldots, n \]

where $i$ is the subscript of the $i$-th rigid body. The Jacobian matrix contains the first derivatives of the position vector with respect to the generalized coordinates

\[ J_i = \left[ \frac{\partial r_i}{\partial q_1}, \frac{\partial r_i}{\partial q_2}, \ldots, \frac{\partial r_i}{\partial q_f} \right] \]

Die LE2 have the general form

\[ \frac{d}{dt} \left( \frac{\partial E_{\text{kin}}}{\partial q_k} \right) - \frac{\partial E_{\text{kin}}}{\partial q_k} = Q_k \quad k = 1, \ldots, f \]  

(3.4.1)

\(^1\) There are also Lagrange’s equations of the first kind, which we do not treat here. These equations contain the constraints explicitely, while the equations of the second kind work with the generalized coordinates which already contain the constraints implicitely.
where $E_{\text{kin}}$ is the total kinetic energy of the whole system which is the sum of the kinetic energies of all $n$ rigid bodies:

$$E_{\text{kin}} = \sum_{i=1}^{n} E_{\text{kin},i} \quad (3.4.2)$$

The equation (3.4.1) yield $f$ differential equations according to the number of the degrees of freedom $f$, respectively the number of generalized coordinates. The $Q_k$ are generalized forces. Which result from the external impressed forces from all $n$ rigid bodies.

The generalized force $Q_k$ follows from the forces and moments ($l$-th force/moment at the $i$-th body)

$$Q_k = \sum_{i=1}^{n} \left( \sum_{l=1}^{L_i} \frac{\partial r_{il}}{\partial q_k} \right)^T \left( \sum_{l=1}^{L_i} \left( \frac{\partial r_{il}}{\partial q_k} \right)^T M_{il}^{(e)} \right)$$

by a projection of a force or moment with the help of the Jacobian matrix. This is done in such a way that only the part of the force/moment is considered that contributes to the motion of the system (to the $k$-th gen. coordinate) according to the constraints (example is given in the class room).

### 3.4.1 Conservative Systems

Generally, for conservative forces a potential exists (see chapter 3.1.4.2), which is not the case for non-conservative forces. Thus, if the generalized force results from conservative forces/moments we can derive them directly from the potential energy

$$Q_k = Q_{k,c} = -\frac{\partial E_{\text{pot}}}{\partial q_k} \quad (3.4.4)$$

As shown earlier, conservative forces can be calculated from the potential by differentiation of $E_{\text{pot}} = E_{\text{pot}}(q_1, \ldots, q_f)$ but now with respect to the $k$-th generalized coordinate. Introducing eqn.3.4.4 into (3.4.1) we get

$$\frac{d}{dt} \left( \frac{\partial E_{\text{kin}}}{\partial q_k} \right) - \frac{\partial E_{\text{kin}}}{\partial q_k} = -\frac{\partial E_{\text{pot}}}{\partial q_k} \quad (3.4.5)$$

with the total energies $E_{\text{kin}} = \sum_{i=1}^{n} E_{\text{kin},i}$ and $E_{\text{pot}} = \sum_{i=1}^{n} E_{\text{pot},i}$

Introducing the Lagrangian function $L^2$ (also called kinetic potential)

$$L = E_{\text{kin}} - E_{\text{pot}} \quad (3.4.6)$$

\[2\] A confusion of the Lagrangian function $L$ with the angular momentum $L$ is not expected here
we see that

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \quad \text{for} \quad k = 1, \ldots, f
\]  

(3.4.7)

which are the famous Langrangian equations of the 2. kind.

**3.4.2 Conservative and Non-conservative Forces, Rayleigh Energy Dissipation Function**

In this case a potential exists only for the conservative forces/moments and only these forces/moments can be treated by potential energy terms. The non-conservative forces/moments have to be considered by the generalized forces

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_{k,nc} \quad \text{for} \quad k = 1, \ldots, f
\]  

(3.4.8)

However, some types of non-conservative forces/moments (like viscous damping forces) can be expressed by an energy dissipation function \(D\) which has the dimension of a power. This dissipation function is called *Rayleigh dissipation function* and can be defined by a general expression

\[
D = \frac{1}{2} \sum_i \sum_j c_{ij} \dot{q}_i \dot{q}_j
\]  

(3.4.9)

where the the \(\dot{q}'s\) are the time derivatives of the generalized coordinates \(q\) and the \(c's\) are coefficients. The non-conservative forces can be derived from \(D\) in a similar way as the conservative forces, namely by differentiating a scalar function:

\[
Q_{k,nc} = -\frac{\partial D}{\partial \dot{q}_k}
\]  

(3.4.10)

but here we differentiate with respect to a velocity. This leads to an extension of the Lagrangian equations

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_{k,nc} - \frac{\partial D}{\partial \dot{q}_k}
\]  

(3.4.11)

Here the \(Q's\) only contain those forces which cannot be expressed by \(D\).

As an example we can consider a viscous damper with damping constant \(c\). The motion of the left attachment point of the damper is described by \(q_l\) and the right attachment point by \(q_r\), both having the same direction. The dissipation function for this case is

\[
D = \frac{1}{2} c v_{rel}^2 = \frac{1}{2} c (q_r - q_l)^2
\]
As we can see this example also leads to the general form of eqn.(3.4.9):

\[ D = \frac{1}{2} c (\dot{q}_r - \dot{q}_l)^2 = \frac{1}{2} c (\ddot{q}_r \dot{q}_r - 2\ddot{q}_l \dot{q}_r + \ddot{q}_l \dot{q}_l) \]

The damping force is obtained by differentiation

\[ F_{\text{damper},r} = -\frac{\partial D}{\partial \dot{q}_r} = -c (\dot{q}_r - \dot{q}_l) \]

\[ F_{\text{damper},l} = -\frac{\partial D}{\partial \dot{q}_l} = c (\dot{q}_r - \dot{q}_l) \]

### 3.5 Equations of Motion of a Mechanical System

From the Lagrangian equations or Newton/Euler’s equations we get the differential equations describing the motion of a mechanical system with \( f \) dof. They have the general form

\[ M(q) \ddot{q} + h(q, \dot{q}, t) = Q(t) \]  

(3.5.1)

where

- \( M(q) \): \( f \times f \) mass matrix (which in general is depending on the generalized coordinates \( q \))
- \( Q(t) \): vector of time dependent external forces and moments (dimension \( f \))
- \( h(q, \dot{q}, t) \): vector of forces/moments (dimension \( f \)) depending on the generalized coordinates and/or the generalized velocities and/or time \( t \), e.g.
  - Conservative elastic forces, depending on \( q \)
  - Dissipative forces, e.g. from viscous damping, friction, depending on \( \dot{q} \)
  - Gyroscopic moments, depending on velocities \( \dot{q} \)
  - …..

#### 3.5.1 Linearization of the Equations of Motion

In many applications, the motion described by \( q(t) \) can be split into a reference motion \( q_r(t) \) which is the desired motion of the system and a more or less small disturbance \( \Delta q(t) \) (e.g. a vibration about this reference motion \( q_r(t) \)).

\[ q(t) = q_r(t) + \Delta q(t) \]  

(3.5.2)
The same can be done with the external forces which are split into reference forces and disturbances

\[ \mathbf{Q}(t) = \mathbf{Q}_r(t) + \Delta \mathbf{Q}(t) \]  \hspace{1cm} (3.5.3)

With

\[ F(\ddot{q}, \dot{q}, q, t) = M(q) \ddot{q} + h(q, \dot{q}, t) = \mathbf{Q}(t) \]  \hspace{1cm} (3.5.4)

we can derive a linearized differential equation for the motion around the desired trajectory \( q_d(t) \) by using the Taylor series which we truncate after the linear part:

\[
F(\ddot{q}_r + \Delta \ddot{q}_r, \dot{q}_r + \Delta \dot{q}_r, q_r + \Delta q_r, t) = F(\ddot{q}_r, \dot{q}_r, q_r, t) + \frac{\partial F}{\partial \ddot{q}} \Delta \ddot{q}_r + \frac{\partial F}{\partial \dot{q}} \Delta \dot{q}_r + \frac{\partial F}{\partial q} \Delta q_r
\]

Splitting the last equation into reference motion (which can also be a static displacement) and disturbance (which is a small motion around the reference trajectory), we get

1) Reference motion

\[ M(q_r) \ddot{q}_r + h(q_r, \dot{q}_r, t) = \mathbf{Q}_r(t) \]  \hspace{1cm} (3.5.5)

2) Disturbance

\[
M(q_r) \Delta \ddot{q}_r + \frac{\partial h}{\partial \dot{q}} \Delta \dot{q}_r + \frac{\partial h}{\partial q} \Delta q_r + \frac{\partial (M(q)\ddot{q}_s)}{\partial q} \Delta q = \Delta \mathbf{Q}(t)
\]  \hspace{1cm} (3.5.6)

3.5.2 Equation of Motion of a Linear Time-Variant and Time-Invariant Mechanical System

If the linearization is done with respect to a reference trajectory \( q_d(t) \) the resulting system matrices are time dependent. We call this a time-variant system.

\[ M(t) \Delta \ddot{q} + (C(t) + G(t)) \Delta \dot{q} + (K + N) \Delta q = \Delta \mathbf{Q}(t) \]  \hspace{1cm} (3.5.7)
If the linearization is done with respect to a reference point \( q_r \) (e.g. the static equilibrium position due to the gravitational forces) the resulting system matrices are time independent and the system is called \textit{time-invariant}. The system matrices are

\[
M \Delta \ddot{q} + (C + G) \Delta \dot{q} + (K + N) \Delta q = \Delta Q(t)
\] (3.5.8)

The \( f \times f \) matrices of the equation of motion are:

- \( M \): mass matrix
- \( C \): symmetric damping matrix (describing velocity dependent damping forces/moments)
- \( G \): skew-symmetric gyroscopic matrix (describing velocity dependent gyroscopic moments)
- \( K \): symmetric stiffness matrix (describing position dependent restoring forces/moments)
- \( N \): skew-symmetric matrix of non-conservative position dependent forces/moments

### 3.6 State Space Representation of a Mechanical System

#### 3.6.1 The General Non-linear Case

The equation of motion

\[
M(q) \ddot{q} + h(q, \dot{q}, t) = Q(t)
\] (3.5.1)

is in general nonlinear and contains the second order derivatives with respect to time as highest order (due to the fact the acceleration appear in the equations). However, many numerical algorithms\(^3\) for ordinary differential equations (ODE) and theories in control make use of first-order formulations.

\[
\dot{z} = f(z, t)
\] (3.6.1)

which is a first-order \textit{differential equation}. The second-order equation of motion (3.5.1) has to be converted into a first-order equation which is done by the following intermediate step. We introduce the \textit{state space vector} \( \mathbf{z} \):

\[
\mathbf{z} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\] (3.6.2)

---

\(^3\): Examples are the Euler- (very simple but inaccurate algorithm), the Runge-Kutta- or the Adams-method
which contains the generalized coordinates and the velocities as well. The state of a mechanical system is defined by the generalized coordinates (displacements or angles) and the velocities!

The acceleration comes into the play when the derivative of (3.6.2) is formulated:

\[
\dot{\mathbf{z}} = \begin{pmatrix} \dot{q} \\ \dot{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix}
\]  \hspace{1cm} (3.6.3)

Comparing (3.6.3) and (3.6.2), we see that

\[
\dot{z}_1 = \dot{z}_2
\]  \hspace{1cm} (3.6.4)

If we use \( z \) instead of the \( q' \)'s, (3.5.1) becomes

\[
M(z_1) \dot{z}_2 + h(z_1, z_2, t) = Q(t)
\]  \hspace{1cm} (3.6.5)

so that we only have first-order time derivatives. However we have to pay the price that the number of the differential equations double and has dimension \( 2f \) now.

Converting eqns.(3.6.4) and (3.6.5) into (3.6.1) we get:

\[
\dot{\mathbf{z}} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \dot{z}_2 \\ M^{-1}(z_1) \left[ Q(t) - h(z_1, z_2, t) \right] \end{pmatrix} = f(z, t)
\]  \hspace{1cm} (3.6.6)

For the solution of the problem we also have to define the initial position and the initial velocities:

\[
z(t = 0) = z_0 = \begin{pmatrix} q_0 \\ \dot{q}_0 \end{pmatrix}
\]  \hspace{1cm} (3.6.7)

Usually, in a state-space representation, a second equation, the measurement equation, is used. The measurement equation links the measured quantities \( y \) with state-space variables in \( z \):

\[
y = g(z, t)
\]  \hspace{1cm} (3.6.8)

This is a very general (non-linear) formulation. As an example a measured quantity \( y_1 \) could be a strain which has to be related to displacements \( q \).
3.6.2 The Linear Time-Invariant Case

In many practical cases it is possible to linearize the non-linear equations of motion or they are linear from the beginning. The linear theory is well examined and many important theorems are available for linear systems.

If the mechanical system behaves linearly, the equation of motion is (the ∆ is left away here for simplicity)

\[ M \ddot{q} + (C+G) \dot{q} + (K+N)q = Q(t) \]  

(3.6.9)

In this case the state-space equation \( z = f(z,t) \) becomes

\[ \dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}(K+N) & -M^{-1}(C+G) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}Q(t) \end{pmatrix} \]

(3.6.10)

This can be re-organized in the following way:

\[ \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}(K+N) & -M^{-1}(C+G) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}Q(t) \end{pmatrix} \]

(3.6.11)

or

\[ \begin{pmatrix} \dot{q} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}(K+N) & -M^{-1}(C+G) \end{pmatrix} \begin{pmatrix} q \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}Q(t) \end{pmatrix} \]

(3.6.12)

The general formulation of a linear state space model in control theory has the form

\[ \dot{z} = Az + Bu \]

(3.6.13)

where

\[ A : 2f \times 2f \text{ system matrix} \]
\[ B : 2f \times 1 \text{ input matrix} \]
\[ u: \text{ input (here the generalized forces } u(t) = Q(t)) \]

The two matrices can be identified as:

\[ A = \begin{pmatrix} 0 & I \\ -M^{-1}(K+N) & -M^{-1}(C+G) \end{pmatrix} \]

(3.6.14)
Assume we have measured \( n_y \) quantities which we arrange in the vector \( y \). The corresponding linear measurement equation is

\[
y = C_{\text{meas}} z + D_{\text{meas}} u(t)
\]  
(3.6.15)

with

\[
C_{\text{meas}}: \text{Measurement or Output matrix (size } n_y \times 2f) \\
D_{\text{meas}}: \text{Transmission matrix (size } n_y \times f)
\]

The eigenvalues \( \lambda \) of the system matrix \( A \) are the system poles and they contain important information about the resonant frequencies and damping behaviour. If one of the real parts of the \( \lambda \)'s is positive, the system is unstable and large amplitude can occur so that the system can be destroyed. More about these relations will be given in the chapters dealing with vibrations of mechanical systems.

3.6.3 The Linear Time-Invariant Case in Discrete Time

Dealing with digital techniques, we have sampled time series and the values are only available at time instants \( t_k, k = 1, 2, 3, \ldots \).

Assuming that \( u \) is constant between two sample times and transforming eq.3 from discrete time to continuous time yields:

\[
z(k + 1) = A_{\text{dis}} z(k) + B_{\text{dis}} u(k)
\]  
(3.6.16)

The measurement equation can be written as

\[
y(k) = C_{\text{meas}} z(k) + D_{\text{meas}} u(k)
\]  
(3.6.17)

where the discrete-time matrices can be calculated from the continuous-time matrices \( A \) and \( B \):

\[
A_{\text{dis}} = e^{A \cdot \Delta t}
\]  
(3.6.18)

\[
B_{\text{dis}} = \int_{0}^{\Delta t} e^{A \cdot \tau} d \tau B
\]  
(3.6.19)

This can be shown by solving the differential equation\(^4\). The matrices of the measurement equations remain unchanged, because the measurement equation is an algebraic equation and contains no differential expression.