5 Vibration of Linear Multiple-Degree-of-Freedom Systems

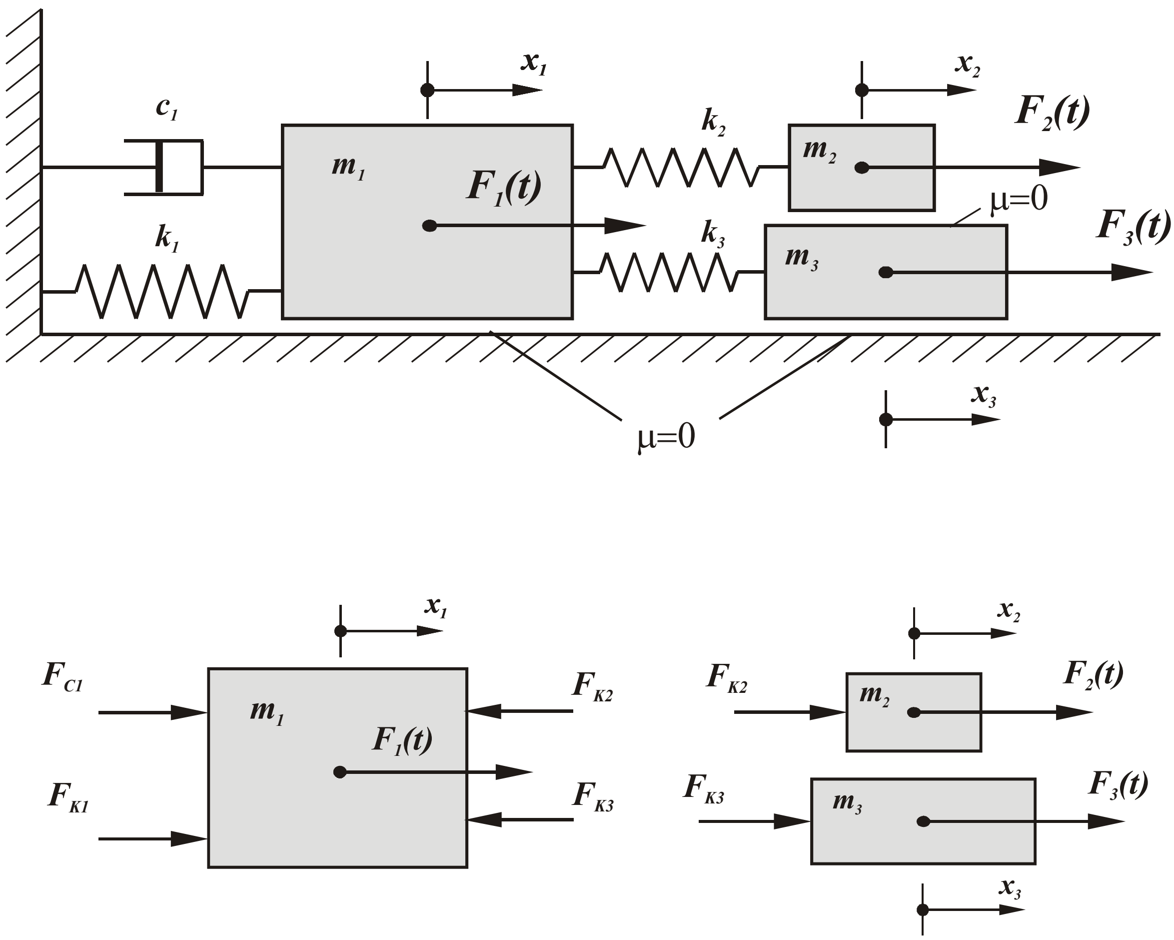


Fig. 5.1: Multi-Degree-of-Freedom System with Free Body Diagram

5.1 Equation of Motion

The equation of motion can be derived by using the principles we have learned such as Newton's/Euler's laws or Lagrange's equation of motion. For a general linear system mdof system we found that we can write in matrix form

$$\boxed{\underline{\underline{M}}\,\ddot{\underline{x}} + \left(\underline{\underline{C}} + \underline{\underline{G}}\right)\dot{\underline{x}} + \left(\underline{\underline{K}} + \underline{\underline{N}}\right)\underline{x} = \underline{F}} \qquad (5.1.1)$$

with the matrices

$\underline{\underline{M}}$: Mass matrix (symmetric) $\underline{\underline{M}} = \underline{\underline{M}}^T$

$\underline{\underline{C}}$: Damping matrix (symmetric) $\underline{\underline{C}} = \underline{\underline{C}}^T$

$\underline{\underline{K}}$: Stiffness matrix (symmetric) $\underline{\underline{K}} = \underline{\underline{K}}^T$

$\underline{\underline{G}}$: Gyroskopic matrix (skew-symmetric) $\underline{\underline{G}} = -\underline{\underline{G}}^T$

$\underline{\underline{N}}$: Matrix of non-conservative forces (skew-symmetric) $\underline{\underline{N}} = -\underline{\underline{N}}^T$

$\underline{F}$: External forces

Note:
A general matrix A can be decomposed into the symmetric part and the skew-symmetric part by the following manipulation:

$$\underline{\underline{A}} = \underbrace{\underbrace{\frac{1}{2}\left(\underline{\underline{A}} + \underline{\underline{A}}^T\right)}_{symmetric} + \underbrace{\frac{1}{2}\left(\underline{\underline{A}} - \underline{\underline{A}}^T\right)}_{skew-symmetric}}_{part}$$

In the standard case that we have no gyroscopic forces and no non-conservative displacement dependent forces but only inertial forces, damping forces and elastic forces the last equation reduces to

$$\boxed{\underline{\underline{M}}\,\underline{\ddot{x}} + \underline{\underline{C}}\,\underline{\dot{x}} + \underline{\underline{K}}\,\underline{x} = \underline{F}} \tag{5.1.2}$$

Example:
The system shown in fig. 5.1, where the masses can slide without friction ($\mu = 0$), has the following equation of motion.

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 + k_3 & -k_2 & -k_3 \\ -k_2 & k_2 & 0 \\ -k_3 & 0 & k_3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{pmatrix} \tag{5.1.3}$$

5.2 Influence of the Weight Forces and Static Equilibrium

The static equilibrium displacements are calculated by ($\underline{\ddot{x}}_{stat} = \underline{\dot{x}}_{stat} = \underline{0}$):

$$\underline{\underline{K}}\,\underline{x}_{stat} = \underline{F}_{stat} \tag{5.2.1}$$

which in the case of the example shown in fig. 5.2:

$$\underline{\underline{K}}\,\underline{x}_{stat} = \underline{F}_{stat} = \begin{pmatrix} m_1 g \\ m_2 g \end{pmatrix}$$

The dynamic problem for this example is

$$\underline{\underline{M}}\,\ddot{\underline{x}} + \underline{\underline{K}}\,\underline{x} = \underbrace{\underline{F}_{stat}}_{static\ forces} + \underline{F}(t)$$

$$\underline{x} = \underline{x}_{stat} + \underbrace{\underline{x}_{dyn}}_{\substack{part\ of\ the\ motion\ describing\ the\ vibration \\ about\ the\ static\ equilibrium}} \tag{5.2.2}$$

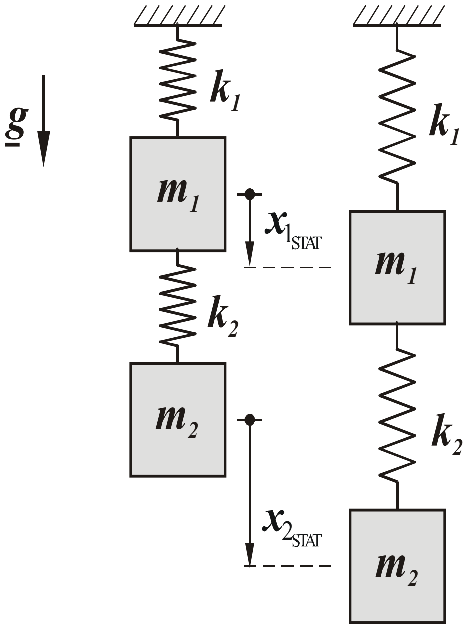


Fig. 5.2: Static equilibrium position of a two dof system

From the last equation also follows that

$$\dot{\underline{x}} = \dot{\underline{x}}_{dyn} \Rightarrow \ddot{\underline{x}} = \ddot{\underline{x}}_{dyn}$$

so that

$$\underline{\underline{M}}\,\ddot{\underline{x}}_{dyn} + \underline{\underline{K}}(\underline{x}_{dyn} + \underline{x}_{stat}) = \underline{F}_{stat} + \underline{F}(t) \tag{5.2.3}$$

and after rearrangement

$$\underline{\underline{M}}\,\ddot{\underline{x}}_{dyn} + \underline{\underline{K}}\,\underline{x}_{dyn} = \underbrace{\underline{F}_{stat} - \underline{\underline{K}}\,\underline{x}_{stat}}_{=0} + \underline{F}(t) \tag{5.2.4}$$

$$\underline{\underline{M}}\,\ddot{\underline{x}}_{dyn} + \underline{\underline{K}}\,\underline{x}_{dyn} = F(t) \tag{5.2.5}$$

As can be seen the static forces and static displacements can be eliminated and the equation of motion describes the dynamic process about the static equilibrium position.

Note:
In cases where the weight forces influences the dynamic behavior a simple elimination of the static forces and displacements is not possible. In the example of an inverted pendulum shown in fig. 5.3 the restoring moment is $mgl\,\sin\phi$, where l is the length of the pendulum.

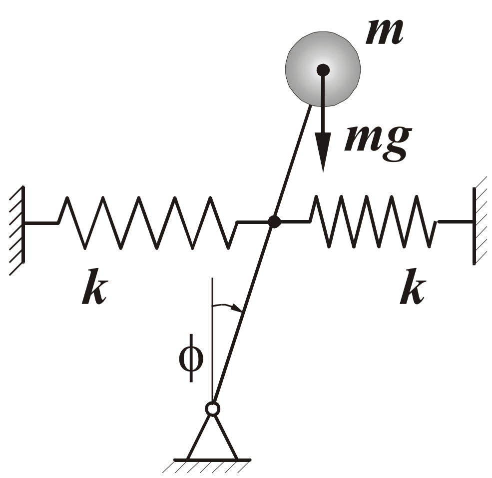


Fig. 5.3: Case where the static force also influences the dynamics

5.3 Ground Excitation

Fig. 5.4 shows a mdof system

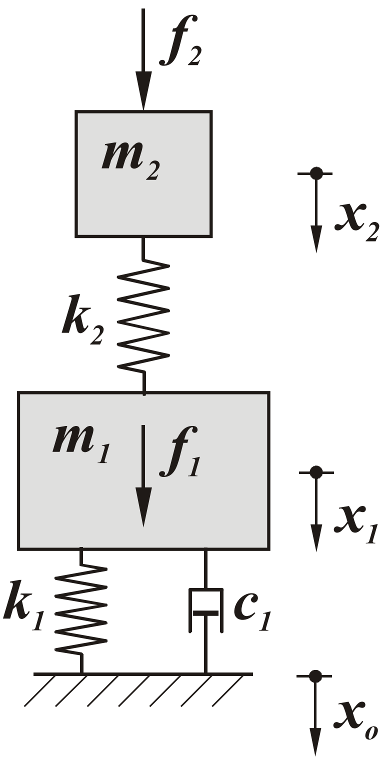


Fig. 5.4: 2dof System with excitation by ground motion x_0

Without ground motion $x_0 = 0$ the equation of motion is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{5.3.1}$$

Now, if we include the ground motion, the differences (x_1-x_0) and the relative velocity d(x_1-x_0)/dt determine the elastic and the damping force, respectively at the lower mass. This can be expressed by adding x_0 to the last equation in the following manner

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}+\begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}+\begin{bmatrix} k_1 \\ 0 \end{bmatrix}x_0(t)+\begin{bmatrix} c_1 \\ 0 \end{bmatrix}\dot{x}_0(t) \tag{5.3.2}$$

The dynamic force f_0 of the vibrating system on the foundation is

$$f_0 = k_1(x_1 - x_0) + c_1(\dot{x}_1 - \dot{x}_0) \tag{5.3.3}$$

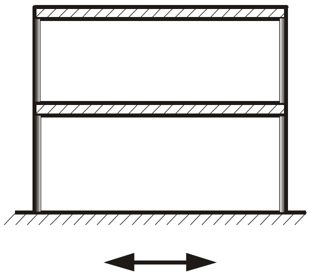


Fig. 5.5: Example for ground motion excitation of a building structure (earthquake excitation)

Fig. 5.6: Excitation of a vehicle by rough surface

5.4 Free Undamped Vibrations of the Multiple-Degree-of-Freedom System

5.4.1 Eigensolution, Natural Frequencies and Mode Shapes of the System

The equation of motion of the undamped system is

$$\boxed{\underline{\underline{M}}\,\underline{\ddot{x}} + \underline{\underline{K}}\,\underline{x} = \underline{0}} \tag{5.4.1}$$

To find the solution of the homogeneous differential equation, we make the harmonic solution approach as in the sdof case. However, now we have to consider a distribution of the individual amplitudes for each coordinate. This is done by introducing (the unknown) vector φ:

$$\underline{x} = \underline{\varphi}\, e^{i\omega t}$$
$$\underline{\ddot{x}} = -\omega^2 \underline{\varphi}\, e^{i\omega t} \qquad (5.4.2)$$

Putting this into eqn. (5.4.1) yields

$$\boxed{\left(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}\right)\underline{\varphi} = \underline{0}} \qquad (5.4.3)$$

This is a homogeneous equation with unknown scalar ω and vector $\underline{\varphi}$. If we set $\lambda = \omega^2$ we see that this is a general matrix *eigenvalue problem*[1]:

$$\boxed{\left(\underline{\underline{K}} - \lambda \underline{\underline{M}}\right)\underline{\varphi} = \underline{0}} \qquad (5.4.4)$$

where λ is the eigenvalue and $\underline{\varphi}$ is the eigenvector. Because the dimension of the matrices is *f* by *f* we get f pairs of eigenvalues and eigenvectors:

$$\boxed{\lambda_i = \omega_i^{\,2} ....i = 1,2..., f} \qquad (5.4.5)$$

ω_i is the i-th natural circular frequency and

$\underline{\varphi}_i$ the i-th eigenvector which has the physical meaning of a vibration mode shape

The solution of the characteristic equation

$$\boxed{\det\left(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}\right) = 0} \qquad (5.4.6)$$

yields the eigenvalues and natural circular frequencies $\lambda = \omega^2$, respectively. The natural frequencies are:

$$\boxed{f_i = \frac{\omega_i}{2\pi}} \qquad (5.4.7)$$

The natural frequencies are the resonant frequencies of the structure.

The eigenvectors can be normalized arbitrarily, because they only represent a vibration mode shape, no absolute values. Commonly used normalizations are

1) Normalize $\underline{\varphi}_i$ so that $\left|\underline{\varphi}_i\right| = 1$
2) Normalize $\underline{\varphi}_i$ so that the maximum component is 1.
3) Normalize $\underline{\varphi}_i$ so that the modal mass (the generalized mass) is 1.

Generalized mass or modal mass: $$\boxed{M_i = \underline{\varphi}_i^{\,T}\, \underline{\underline{M}}\, \underline{\varphi}_i} \qquad (5.4.8)$$

[1] The well-known special eigenvalue problem has the form $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \underline{0}$, where I is the identity matrix, $\underline{x}$ the eigenvector and λ the eigenvalue.

Generalized stiffness: $$\boxed{K_i = \underline{\varphi}_i^T \underline{\underline{K}} \underline{\varphi}_i} \tag{5.4.9}$$

$$\boxed{\begin{aligned} &K_i = \underline{\varphi}_i^T \underline{\underline{K}} \underline{\varphi}_i = \omega_i^2 \\ &if: M_i = 1 \end{aligned}} \tag{5.4.10}$$

The so-called *Rayleigh ratio* is

$$\boxed{\omega_i^2 = \frac{\underline{\varphi}_i^T \underline{\underline{K}} \underline{\varphi}_i}{\underline{\varphi}_i^T \underline{\underline{M}} \underline{\varphi}_i}} \tag{5.4.11}$$

It allows the calculation of the frequency if the vectors are already known.

5.4.2 Modal Matrix, Orthogonality of the Mode Shape Vectors

If we order the natural frequencies so that

$$\omega_1 \le \omega_2 \le \omega_3 \le \ldots \le \omega_f$$

and put the corresponding eigenvectors columnwise in a matrix, the so-called modal matrix, we get

Modal Matrix: $$\boxed{\underline{\underline{\Phi}} = \left[\underline{\varphi}_1, \underline{\varphi}_2, \ldots, \underline{\varphi}_f\right] = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1f} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2f} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{f1} & \varphi_{f2} & \cdots & \varphi_{ff} \end{bmatrix}} \tag{5.4.12}$$

The first subscript of the matrix elements denotes the no. of the vector component while the second subscript characterizes the number of the eigenvector.

The eigenvectors are linearly independent and moreover they are orthogonal. This can be shown by a pair *i* and *j*

$$\left(\underline{\underline{K}} - \omega_i^2 \underline{\underline{M}}\right)\underline{\varphi}_i = \underline{0} \quad \text{and} \quad \left(\underline{\underline{K}} - \omega_j^2 \underline{\underline{M}}\right)\underline{\varphi}_j = \underline{0} \tag{5.4.13}$$

Premultplying by the transposed eigenvector with index *j* and *i* respectively:

$$\varphi_j^T \left(\underline{\underline{K}} - \omega_i^2 \underline{\underline{M}}\right)\underline{\varphi}_i = 0 \quad \text{and} \quad \varphi_i^T \left(\underline{\underline{K}} - \omega_j^2 \underline{\underline{M}}\right)\underline{\varphi}_j = 0 \tag{5.4.14}$$

If we take the transpose of the second equation:

$$\varphi_j^T \left(\underline{\underline{K}}^T - \omega_j^2 \underline{\underline{M}}^T\right)\underline{\varphi}_i = 0 \tag{5.4.15}$$

and consider the symmetry of the matrices: $\underline{\underline{M}} = \underline{\underline{M}}^T$ and $\underline{\underline{K}} = \underline{\underline{K}}^T$ and subtract this equation $\varphi_j^T \left(\underline{\underline{K}} - \omega_j^2 \underline{\underline{M}} \right) \underline{\varphi}_i = \underline{0}$ from the first equation (5.4.14) we get

$$\left(\omega_j^2 - \omega_i^2 \right) \varphi_j^T \left(\underline{\underline{M}} \right) \underline{\varphi}_i = 0 \qquad (5.4.16)$$

which means that if the eigenvalues are distinct $\omega_i \neq \omega_j$ for $i \neq j$ the second scalar product expression must be equal to zero:

$$\varphi_j^T \underline{\underline{M}} \underline{\varphi}_i = 0 \qquad (5.4.17)$$

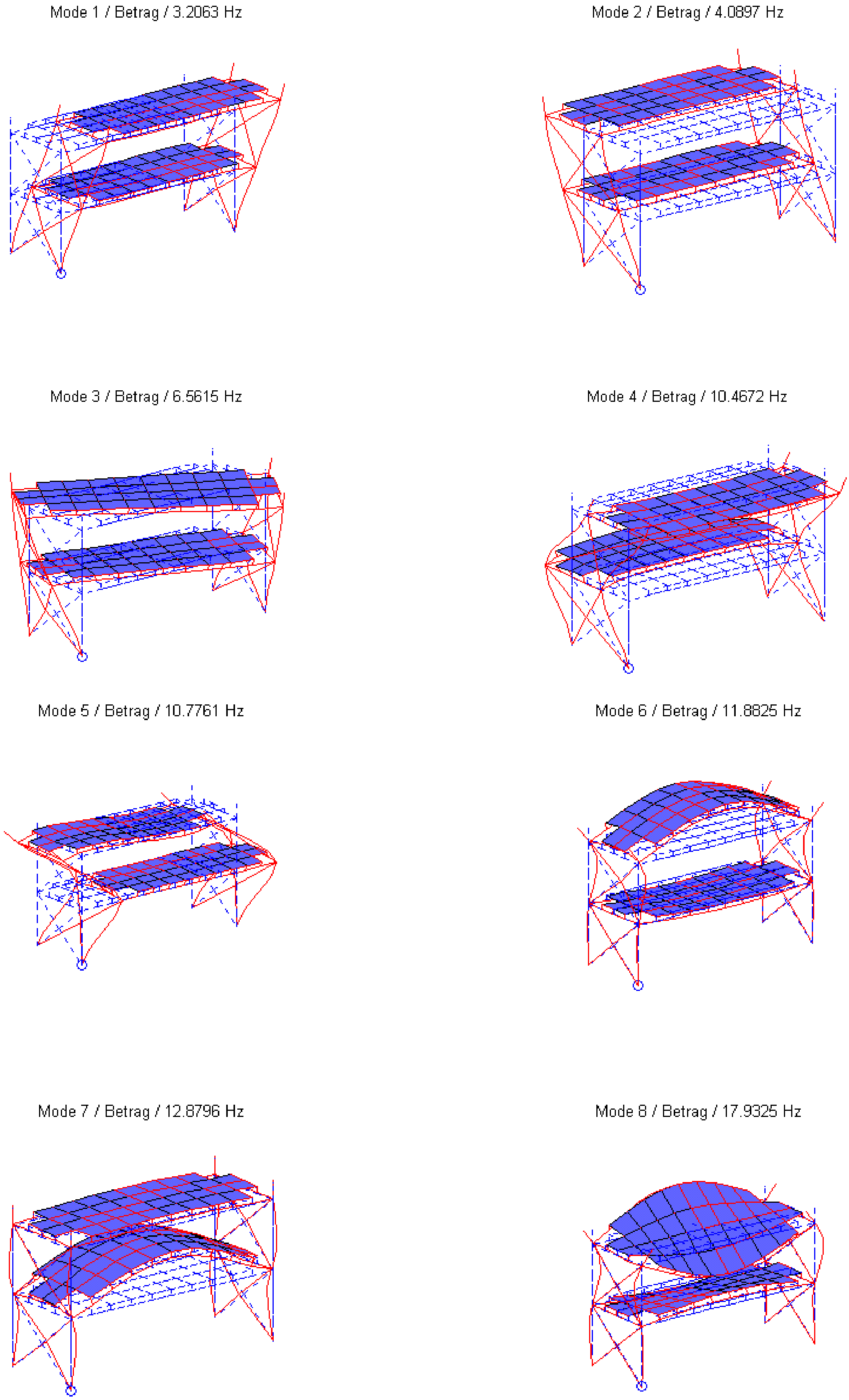
That means that the two distinct eigenvectors $i \neq j$ are *orthogonal with respect to the mass matrix*. For all combinations we can write:

$$\left.\begin{aligned} \underline{\varphi}_j^T \underline{\underline{M}} \underline{\varphi}_i &= \delta_{ij} M_i \\ \underline{\varphi}_j^T \underline{\underline{K}} \underline{\varphi}_i &= \delta_{ij} \omega_i^2 M_i \end{aligned}\right\} \delta_{ij} = \begin{cases} 1 & for \quad i = j \\ 0 & for \quad i \neq j \end{cases} \text{............} Kronecker - Symbol \qquad (5.4.18)$$

or with the modal matrix:

$$\underline{\underline{\Phi}}^T \underline{\underline{M}} \underline{\underline{\Phi}} = diag\{M_i\} = \begin{bmatrix} M_1 & & & \\ & M_2 & & \underline{\underline{0}} \\ \underline{\underline{0}} & & \cdots & \\ & & & M_f \end{bmatrix}$$

$$\underline{\underline{\Phi}}^T \underline{\underline{K}} \underline{\underline{\Phi}} = diag\{\omega_i^2 M_i\} = \begin{bmatrix} \omega_1^2 M_1 & & & \\ & \omega_2^2 M_2 & & \underline{\underline{0}} \\ \underline{\underline{0}} & & \cdots & \\ & & & \omega_f^2 M_f \end{bmatrix} \qquad (5.4.19)$$



Example: Mode shapes and natural frequencies of a two storey structure

5.4.3 Free Vibrations, Initial Conditions

The free motion of the undamped system $\underline{x}$(t)is a superposition of the modes vibrating with the corresponding natural frequency:

$$\underline{x}(t) = \sum_{i=1}^{f} \underline{\varphi}_i \left(A_{ci} \cos \omega_i t + A_{si} \sin \omega_i t \right) \tag{5.4.20}$$

Each mode is weighted by a coefficient A_{ci} and A_{si} which depend on the initial displacement shape and the velocities. In order to get these coefficients, we premultiply (5.4.20) by the transposed j-th eigenvector:

$$\underline{\varphi}_j^T M \underline{x}(t) = \underbrace{\sum_{i=1}^{f} \underline{\varphi}_j^T M \underline{\varphi}_i \left(A_{ci} \cos \omega_i t + A_{si} \sin \omega_i t \right)}_{f\ddot{u}r.i \neq j \Rightarrow = 0} = \underbrace{\underline{\varphi}_j^T M \underline{\varphi}_j}_{M_j} \left(A_{cj} \cos \omega_j t + A_{sj} \sin \omega_j t \right) \tag{5.4.21}$$

All but one of the summation terms are equal to zero due to the orthogonality conditions. With the initial conditions for $t = 0$ we can derive the coefficients:

$$t = 0$$
$$\underline{x}(t = 0) = \underline{x}_0$$
$$\underline{\varphi}_j^T M \underline{x}_0 = M_j A_{cj}$$

$$\boxed{A_{cj} = \frac{\underline{\varphi}_j^T \underline{\underline{M}} \underline{x}_0}{M_j}} \tag{5.4.22}$$

$$\underline{\dot{x}}(t) = \sum_{i=1}^{f} \underline{\varphi}_i \omega_i \left(-A_{ci} \sin \omega_i t + A_{si} \cos \omega_i t \right)$$
$$t = 0$$
$$\underline{\dot{x}}(t = 0) = \underline{v}_0$$
$$\underline{\varphi}_j^T M \underline{\dot{x}}_0 = M_j \omega_j A_{sj}$$

$$\boxed{A_{sj} = \frac{\underline{\varphi}_j^T \underline{\underline{M}} \underline{v}_0}{M_j \omega_j}} \tag{5.4.23}$$

which we have to calculate for modes j .

5.4.4 Rigid Body Modes

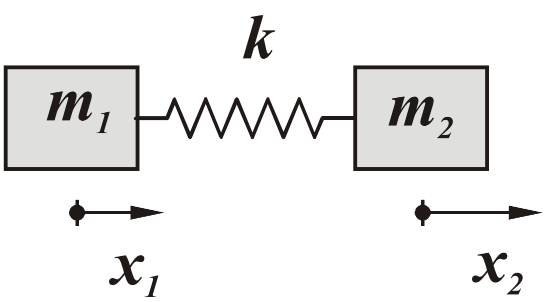


Fig. 5.7: A two-dof oscillator which can perform rigid body motion

As learned earlier the constraints reduce the dofs of the rigid body motion. If the number of constraints is not sufficient to suppress rigid body motion the system has also zero eigenvalues. The number of zero eigenvalues corresponds directly to the number of rigid body modes. In the example shown in Fig. 5.7 the two masses which are connected with a spring can move with a fixed distance as a rigid system. This mode is the rigid body mode, while the vibration of the two masses is a deformation mode.

The equation of motion of this system is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The corresponding eigenvalue problem is

$$\begin{bmatrix} k - \lambda m_1 & -k \\ -k & k - \lambda m_2 \end{bmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The eigenvalues follow from the determinant which is set equal to zero:

$$\det[\cdots] = \left[(k - \lambda m_1)(k - \lambda m_2) - k^2\right] = 0$$

$$\lambda^2 m_1 m_2 - \lambda(k m_1 + k m_2) = 0$$

Obviously, this quadratic equation has the solution

$$\lambda_1 = \omega_1^2 = 0$$

and

$$\lambda_2 = \omega_2^2 = k \frac{m_1 + m_2}{m_1 m_2}$$

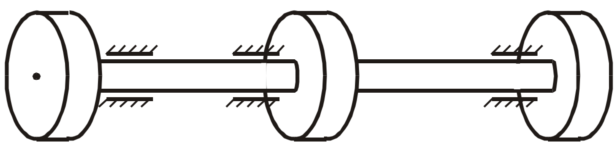
The corresponding (unnormalized) eigenvectors are

$$\underline{\varphi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which is the *rigid body mode*: both masses have the same displacement, no potential energy is stored in the spring and hence no vibration occurs. The second eigenvector, the *deformation mode* is

$$\underline{\varphi}_2 = \begin{pmatrix} 1 \\ -\dfrac{m_1}{m_2} \end{pmatrix}$$

which is a vibration of the two masses. Other examples for systems with rigid body modes are shown in the following figures.



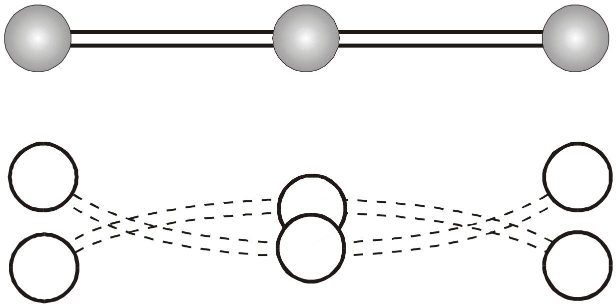


Fig. 5.8: Examples for systems with torsional and transverse bending motion with rigid body motion

Fig. 5.9: Flying airplane (Airbus A318) as a system with 6 rigid body modes and deformation modes

Fig. 5.10: Commercial communication satellite system (EADS) with 6 rigid body modes and deformation modes

5.5 Forced Vibrations of the Undamped Oscillator under Harmonic Excitation

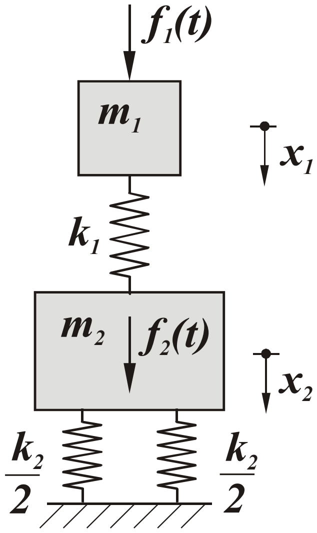


Fig. 5.11: Example for a system under forced excitation

The equation of motion for this type of system is

$$\boxed{\underline{\underline{M}}\,\ddot{\underline{x}} + \underline{\underline{K}}\,\underline{x} = \underline{F}(t)} \qquad (5.5.1)$$

For a harmonic excitation we can make an exponential approach to solve the problem as we did with the sdof system

$$\underline{F}(t) = \underbrace{\hat{\underline{F}}}_{\substack{complex\\ Amplitude\ vector}} e^{i\omega t} \qquad (5.5.2)$$

We make a complex harmonic approach for the displacements with Ω as the excitation frequency:

$$\underline{x} = \hat{\underline{X}} e^{i\Omega t} \qquad (5.5.3)$$

The acceleration vector is the second derivative

$$\ddot{\underline{x}} = -\omega^2 \hat{\underline{X}} e^{i\Omega t} \qquad (5.5.4)$$

Putting both into the equation of motion and eliminating the exp-function yields

$$\boxed{\left(\underline{\underline{K}} - \Omega^2 \underline{\underline{M}}\right)\hat{\underline{X}} = \hat{\underline{F}}} \qquad (5.5.5)$$

which is a complex linear equation system that can be solved by hand for a small number of dofs or numerically. The formal solution is

$$\boxed{\underline{\hat{X}} = \left(\underline{\underline{K}} - \Omega^2 \underline{\underline{M}}\right)^{-1} \underline{\hat{F}}} \qquad (5.5.6)$$

which can be solved if determinant of the coefficient matrix : $\boxed{\det\left(\underline{\underline{K}} - \Omega^2 \underline{\underline{M}}\right) \neq 0}$

If the excitation frequency Ω coincides with one of the natural frequencies ω_i we get resonance of the system with infinitely large amplitudes (in the undamped case)

Resonance: $$\boxed{\det\left(\underline{\underline{K}} - \Omega^2 \underline{\underline{M}}\right) = 0 \quad \Leftrightarrow \quad \Omega = \omega_i}$$