MACHINE DYNAMICS AND SYSTEM DYNAMICS

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Lecture Notes
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# CONTENTS

i  **Lecture Notes**  

1  **Introduction**  

1.1  Tasks of Systems and Machine Dynamics  

1.2  Modelling of Mechanical Systems  

1.2.1  Degree Of Freedom  

1.2.2  Different Categories of Models  

2  **Kinematics**  

2.1  Kinematics of Particles  

2.1.1  Motion on a Straight Path  

2.1.2  Description of the Motion Using Generalized Coordinates  

2.2  Kinematics of Rigid Bodies  

2.2.1  Transformation Matrices  

2.2.2  Velocity in the Inertial System  

2.2.3  Relation Between Matrix and Vector Representation of the Velocity  

2.2.4  Velocity in the Body-Fixed Reference Frame  

2.2.5  Accelerations in the Inertial System  

2.2.6  Acceleration in the Body-Fixed Reference Frame  

2.2.7  Angular Accelerations  

2.2.8  Systems with Constraints  

2.3  Relative Motion of a Particle  

2.3.1  Relation Between Absolute and Relative Velocity  

2.3.2  Relation Between Absolute and Relative Acceleration  

2.3.3  Summary of the Formula for Relative Kinematics  

3  **Kinetics**  

3.1  Kinetics of a Single Particle  

3.1.1  Momentum and Angular Momentum, Newton’s Law  

3.1.2  Rotation of a Body About a Fixed Axis  

3.1.3  Kinetics of a Particle for Relative Motion  

3.1.4  Work and Work-Energy Principles  

3.1.5  Power  

3.2  Kinetics of Rigid Bodies  

3.2.1  Momentum of a Rigid Body and the Momentum Theorem  

3.2.2  Angular Momentum and Moments of Inertia  

3.2.3  Angular Momentum Theorem  

3.2.4  Change of the Reference Frame  


3.2.5 Eulerian Equations, Angular Momentum Theorem in a Rotating Coordinate Frame .......................... 61
3.2.6 Angular Momentum Theorem in a Guided Coordinate System .................................................. 63
3.3 Kinetic Energy of a Rigid Body ................................................. 66
3.4 Lagrange’s Equations of Motion of 2nd Kind ................................. 68
  3.4.1 Conservative Systems .................................................. 69
  3.4.2 Conservative and Non-conservative Forces, Rayleigh Energy Dissipation Function ......................... 70
3.5 Equations of Motion of a Mechanical System ................................. 71
  3.5.1 Linearization of the Equations of Motion ............................. 72
  3.5.2 Equation of Motion of a Linear Time-Variant and Time-Invariant Mechanical System ....................... 73
3.6 State Space Representation of a Mechanical System ....................... 74
  3.6.1 The General Non-linear Case ........................................ 74
  3.6.2 The Linear Time-Invariant Case ...................................... 75
  3.6.3 The Linear Time-Invariant Case in Discrete Time ................. 76
4 Linear Vibrations of Systems with One Degree of Freedom .................. 79
4.1 General Classification of Vibrations ...................................... 79
4.2 Free Undamped Vibrations of the Linear Oscillator ......................... 82
  4.2.1 Equation of Motion .................................................. 82
  4.2.2 Solution of the Equation of Motion ................................ 83
  4.2.3 Complex Notation .................................................. 84
  4.2.4 Relation Between Complex and Real Notation ...................... 85
  4.2.5 Further Examples of Single Degree of Freedom Systems ......... 86
  4.2.6 Approximate Consideration of the Spring Mass ................. 87
4.3 Free Vibrations of a Viscously Damped Oscillator ......................... 89
  4.3.1 Equation of Motion ................................................ 89
  4.3.2 Solution of the Equation of Motion ................................ 91
4.4 Forced Vibrations From Harmonic Excitation ............................... 95
  4.4.1 Excitation with Constant Force Amplitude ......................... 96
  4.4.2 Harmonic Force from Imbalance Excitation ....................... 101
  4.4.3 Support Motion / Ground Motion .................................... 102
4.5 Excitation by Impacts .................................................... 104
  4.5.1 Impact of Finite Duration ......................................... 104
  4.5.2 DIRAC-Impact ..................................................... 106
4.6 Excitation by Forces with Arbitrary Time Functions .................... 107
4.7 Periodic Excitations ..................................................... 108
  4.7.1 Fourier Series Representation of Signals ......................... 108
  4.7.2 Forced Vibration Under General Periodic Excitation .......... 110
4.8 Vibration Isolation of Machines ........................................ 114
4.8.1 Forces on the Environment Due to Excitation by Inertia Forces .......................... 115
4.8.2 Tuning of Springs and Dampers ............................................. 117

5 VIBRATION OF LINEAR MULTIPLE-DEGREE-OF-FREEDOM SYSTEMS 121
5.1 Equation of Motion .......................................................... 121
5.2 Influence of the Weight Forces and Static Equilibrium ................. 122
5.3 Ground Excitation .............................................................. 124
5.4 Free Undamped Vibrations of the Multiple-Degree-of-Freedom System ......................................................... 126
5.4.1 Eigensolution, Natural Frequencies and Mode Shapes of the System ......................................................... 126
5.4.2 Modal Matrix, Orthogonality of the Mode Shape Vectors127
5.4.3 Free Vibrations, Initial Conditions ................................ 130
5.4.4 Rigid Body Modes ............................................................ 130
5.5 Forced Vibrations of the Undamped Oscillator under Harmonic Excitation ......................................................... 132

ii APPENDIX 135
A EINLEITUNG - ERGÄNZUNG 137
B KINEMATICS - APPENDIX 139
  B.1 General 3-D Motion in Cartesian Coordinates ....................... 139
  B.2 Three-Dimensional Motion in Cylindrical Coordinates ............... 140
  B.3 Natural Coordinates, Intrinsic Coordinates or Path Variables ... 141
C FUNDAMENTALS OF KINETICS - APPENDIX 145
  C.1 Special cases for the calculation of the angular momentum .... 145
D VIBRATIONS - APPENDIX 149
  D.1 Excitation with constant amplitude of force - Complex Approach149
  D.2 Excit. with constant amp. of force - Alternative Complex App. 151
  D.3 Fourier Series - Alternative Real Representation ................. 152
  D.4 Fourier Series - Alternative Complex Representation ............ 152
  D.5 Magnification Functions .................................................... 154
E LITERATURE 157
F FORMULARY 159
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>DOF</td>
<td>Degree Of Freedom</td>
<td>6</td>
</tr>
<tr>
<td>SDOF</td>
<td>Single Degree Of Freedom</td>
<td>79</td>
</tr>
<tr>
<td>MDOF</td>
<td>Multiple Degrees Of Freedom</td>
<td>79</td>
</tr>
<tr>
<td>EOM</td>
<td>Equation Of Motion</td>
<td>89</td>
</tr>
<tr>
<td>MBS</td>
<td>Multi-body system</td>
<td>8</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite-Element Method</td>
<td>8</td>
</tr>
</tbody>
</table>
Part I

LECTURE NOTES
INTRODUCTION

1.1 TASKS OF SYSTEMS AND MACHINE DYNAMICS

In system dynamics we are concerned with the prediction and analysis of the evolution of the system’s state. This state can be expressed in terms of temperature pressure, chemical concentrations, electrical currents, wear or damage etc.

Especially in machine and structural dynamics, we deal with the motion in terms of displacements, velocities and accelerations as well as with dynamic internal forces and moments in machine and structures. What we are interested in are e.g. the precision with which a robot can follow a given trajectory, the occurrence of unstable motion, amplitudes of vibration in a stationary state or the transient behaviour following to a disturbance of the system.

Some typical fields of applications can be found in

- automotive engineering (vehicle dynamics, vibration and noise),
- railway systems (high speed train ICE) and magnetic levitation systems (Transrapid),
- space vehicles and satellites,
- airplanes and helicopters,
- robotic systems,
- milling machines,
- printing machines,
- internal combustion engines,
• turbomachinery (steam turbines, water turbines, wind turbines, pumps and turbo compressors → rotor dynamics is a special field of machine dynamics,

• biomechanical systems: walking and mobility prosthetics,

• ....

**Mechatronics**

Other names are used for this special class of problems, including *controlled machines, smart machines, smart structures*, and *intelligent machines*. The term *mechatronics* is mainly in use in Europe and Japan where mechatronic devices such as magnetic bearings or automated cameras have been pioneered. A large-scale application is the mag-lev train that has been developed in Japan and Germany. Active magnetic bearings for small and large rotating machines such as pipeline pumps and machine tool spindles have been developed in Switzerland, France or Japan. In the USA a significant amount of research and development has been directed forward toward *Micro-Electro-Mechanical Systems* or MEMS. Other fields of interest are self-diagnosis of machines and structures using built-in diagnostic devices and computational intelligence as well as vibration and noise control.

The distinguishing feature of most of these systems compared to classical controlled machines has been the incorporation of sensing, actuation, and intelligence in producing and controlling motion in machines and structures. This means that we have to integrate control and intelligence into the mechanical design from the very beginning and not as an add-on after the machine is designed.

**Dynamic Failures**

While dynamic analysis in engineering is often used to create motions in physical systems, in many unwanted dynamic failures are to be avoided. Such failures include:

• large deflections,

---

1 F.C. Moon, *Applied Dynamics*
fatigue of materials from high or low amplitude vibrations,

motion-induced fracture,

dynamic instability, e.g. flutter or chatter,

impact-induced local damage (e.g. delaminations of plies in carbon-fibre reinforced plastics),

motion-induced noise,

instability about a steady motion, e.g. wheels on rails,

thermal heating due to dynamic friction.

**Avoidance of Dynamic Failure**

understand the dynamics before the design becomes a product, using simulation tools and/or measurements,

choose materials with enhanced properties to resist fatigue, fracture or wear, or choose materials with higher damping to minimize resonance,

use passive damping,

use active control,

use internal diagnostics, sensors, limit switches , etc. to detect imminent failure and avoid catastrophe (Structural Health Monitoring (SHM)).

Of great importance is the knowledge of the sources and phenomena of unwanted vibrations. Vibrations may be induced by:

oscillating and rotating machine parts with mass unbalance,

periodic variations of the torque in internal combustion engines,

interaction of mechanical machine parts with a fluid (turbulent wind loads, self-excited vibrations , flutter),

earthquakes (important in civil engineering but also in mechanical engineering in safety relevant areas like nuclear power plants),

road roughness (road vehicles),
1.2 Modelling of Mechanical Systems

1.2.1 Degree Of Freedom

The expression *Degree Of Freedom (DOF)* plays a basic role in the modelling of a system. It is an important quantity to describe the complexity of an analytical or numerical model. In general, an increasing number of DOFs increases the model accuracy, but also computation time and computer memory is increased. The engineer’s art is to find a compromise between accuracy and computational cost. For a mechanical system, the lower limit is given by the number of DOFs of the rigid body motion, the upper limit is infinity (in the case of a continuous distributed parameter system) and can be very high in the case of a very detailed finite-element mesh (e.g. 100,000 DOFs). In practice, for a dynamic analysis, it is recommended to add to the rigid body DOFs as many elastic DOFs so that the highest excitation frequency which is of interest for the technical problem is included in the analysis. If the lowest eigenfrequency is much higher than the highest excitation frequency, then the machine or structure can be modeled as a rigid system and only the DOFs of the rigid bodies are considered.

From “Engineering Mechanics” we know that a free rigid body has 6 DOFs, 3 translational and 3 rotational DOFs, in 3D space, and 3 DOFs in 2D space 2 translational and 1 rotational DOF. A multibody-system of \( n \) rigid bodies, which can move freely without constraints, have

\[
    f_{\text{free}} = \sum_{i=1}^{n} f_{\text{free},i}
\]

where \( f_{\text{free},i} \) is the number of DOFs of the \( i \)-th free body and

\[
    f_{\text{free},i} = \begin{cases} 
    6 & \text{in 3D space} \\
    3 & \text{in 2D space} 
    \end{cases}
\]

No machine would properly work, if the different parts of the machine would not be connected in a certain way, e.g. by hinges, joints etc. These constraints, whose overall number is \( n_c \), reduce the number of the DOFs so that

\[
    f = f_{\text{free}} - n_c
\]
The number \( f \) of the constrained system corresponds to the number of the coordinates which are necessary to describe uniquely the position of the system.

To realize a desired motion in a machine, we need supports, joints, etc. All these elements introduce constraints. According to these constraints we get constraint forces (e.g. in a hinge, holding the two parts of the machine together).

The constraints can be divided into several classes. They can depend on the positions, velocities and sometimes explicitly on time.

Therefore, we distinguish:

**Holonomic** constraints: The constraint equation can be formulated by the positions and time only.

**Non-holonomic** constraints: Besides the positions and time also the velocities appear in the constraint equations.

**Rheonomic** constraints: Time \( t \) appears explicitly in the constraint equation.

**Scleronomic** constraints: Time \( t \) does not appear explicitly.

### 1.2.2 Different Categories of Models

Mechanical systems are characterized by

- inertia effects due to the mass of the single elements of the system,
- elasticity of the elements, in the case that we can neglect the deformations we use a rigid body model.

In addition, we can have effects from:

- damping/friction,
- external forces (from motors, hydraulic elements, or disturbances from the environment).

In some applications we might have also contact or play between some bodies. This leads to nonlinear behavior of the system.
The inertia effects are determined by the mass density distribution and the geometry of the body. In all considerations we keep the mass constant, however, the mass moment of inertia can depend on the actual configuration of the system.

The modelling of elastic elements depends on the ratio of elastic forces/inertia forces, that means whether they can be considered as massless spring elements or as bodies with mass, e.g. the ratio is very large for suspension spring of cars, the vertical motion of the car is very slow compared to the vibrations of the spring. With an idealization of a massless spring, modelling errors will not be too large.

The same arguments hold true for damping elements where the ratio of damping forces/inertia forces play an important role.

The properties of the real technical systems have to be described by idealized models. Basically we distinguish between models with concentrated parameters an distributed parameters. That can be divided into different categories.

**Multibody Systems**

A *Multi-body system (MBS)* consists of rigid bodies, which are subjected to loads in discrete points. Forces and moments can be generated by springs, dampers, external forces (e.g. gravitation) , magnetic forces, devices like motors.

A MBS-model has always a finite number of degrees of freedom. The discrete structure of the model is due to the physical discretization.

**Finite-Element Systems**

A model which is generated by means of the finite-element method is composed of single finite elements which are connected in the so-called nodal points. The properties of the elements are based on distributed elastic and inertial effects. By means of internal functions for each element the properties of the interior of an element can be concentrated on the nodal points.

Standard elements in finite elements programs are e.g. truss, beam, plate, 2D-, or 3D solid elements.

A model based on *Finite-Element Method (FEM)* has a finite number of DOFs (which can become very large). In contrast to MBS, the discretization is reached by *mathematical discretization*. 
Distributed Systems

The system is considered to have a distributed mass and elasticity. The equilibrium of forces and moments are formulated for an infinitesimal small element which leads to partial differential equations which may be difficult to be solved analytically. Distributed systems have an infinite number of DOFs.

There is no discretization and the solution is exact in the sense of the assumptions which are made in continuum mechanics. Here, the solution is a function, not a vector.

The advantage is: the solution is exact, however as disadvantage, the geometries which can be treated usually are very simple (such as beams or plates with constant thicknesses or constant cross sections).

However, it is possible to get approximate solutions using a the classical Ritz-approach or difference methods.

Elastic Multibody Systems

This is a connection of rigid MBS and flexible elastic elements modeled by FEM. We can consider the large motions of MBS which are super imposed by small motions of the elastic parts of the structure. This closes the gap between the two worlds of MBS and FEM.
KINEMATICS

The following chapter is partly a repetition of the material learned in “Engineering Mechanics” and partly a presentation of new materials which are necessary to solve more complicated technical problems, such as the motion of robotic systems.

Kinematics in general deals with the motion of a single particle, rigid bodies or a system of bodies. In “Kinematics” we do not ask how this motion is generated, how large the forces and moments are in links between several bodies etc. This is a task of “Dynamics”. Here, we look only at the geometric relations of single bodies or multibody systems and its behavior in time.

In kinematics the motion of a particle or a rigid body is described by position vectors, as well as the velocities (German: Geschwindigkeit) and accelerations (German: Beschleunigung). We investigate the relations between these quantities which are given by the time derivatives. In general these quantities are vectors.

2.1 KINEMATICS OF PARTICLES

During the motion of a particle with a position $P$ this point is moving on the so-called path (German: Bahnkurve or Bahn). The actual position of $P$ is described uniquely by the position vector $\mathbf{r}(t)$ in a fixed reference coordinate system (German: Inertialsystem). The first derivative of the position vector $\mathbf{r}(t)$ with respect to time yields the velocity vector $\mathbf{v}(t)$ (German: Geschwindigkeitsvektor)

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \dot{\mathbf{r}}(t) .$$

(2.1)

From the second derivative we get the acceleration vector $\mathbf{a}(t)$ (German: Beschleunigungsvektor):

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \mathbf{\ddot{r}}(t) = \frac{d^2\mathbf{r}(t)}{dt^2} = \ddot{\mathbf{r}}(t) .$$

(2.2)
2.1.1 Motion on a Straight Path

The most simple case is the particle motion along a straight line. $s = s(t)$ is the position coordinate along the straight path. We do not need a vector representation for the description for the velocity $v(t)$ and the acceleration $a(t)$ we immediately obtain

$$v(t) = \dot{s}(t)$$

$$a(t) = \ddot{v}(t) = \ddot{s}(t)$$

Given the position depending on time $t$ the velocity and the acceleration can be calculated by differentiation of the function $s(t)$.

In the inverse way, given the acceleration $a(t)$, we obtain the velocity and the position of $P$ by integration in the following way:

$$v(t) = v_0 + \int_{t_0}^{t} a(t^*) \, dt^*$$

$$s(t) = s_0 + \int_{t_0}^{t} v(t^*) \, dt^*$$

where $s_0$ (initial position) and $v_0$ (initial velocity) are called the initial conditions.

In cases where the velocity and the acceleration are not given as a function of time, e.g. $v = v(s)$ or $a = a(v)$; $a = a(s)$ we refer to the literature\(^1\).

---

\(^1\) H.G. Hahn: Technische Mechanik
2.1.2 Description of the Motion Using Generalized Coordinates

Often we face the problem that the position, the velocity and the acceleration of a point $P$ has to be expressed in terms of another set of variables, the generalized coordinates $q(t)$, e.g. in the 3D-space, where the components $q_i$ can be longitudes and angles:

$$q(t) = \begin{bmatrix} q_1(t) & q_2(t) & q_3(t) \end{bmatrix}^T. \quad (2.7)$$

A simple example is the motion of a particle on a circular path. We look for the position, the velocity and the acceleration vector, respectively, in Cartesian coordinates depending on the angle $q(t) = q_1(t) = \alpha(t)$ (example given during the lecture). The solution of this problem can be formalized and solved using computer algebraic systems\(^{2}\). Given a set of generalized coordinates $q(t)$ mit $q(t) \in \mathbb{R}^m; \; 1 \leq m \leq 3$, we first have to establish the relation between the position vector and these generalized coordinates (when treating multibody problem, $m$ can be larger than 3) which has the general form

$$\mathbf{r} = \mathbf{r}(q(t)) \quad (2.8)$$

First, we calculate the velocity and acceleration vector expressed by the generalized coordinates $q_i$. The purpose of this transformation is to express complicated motions of mechanisms or systems like a robot by the rotations of a drive motor, represented by an angle $\alpha(t)$.

2.1.2.1 Velocities

In order to get the velocity $\mathbf{v}(t)$ we differentiate the position vector in eq. (2.8), with respect to time, applying the chain rule. With $m = 3$ we get:

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial q_1} \frac{dq_1}{dt} + \frac{\partial \mathbf{r}}{\partial q_2} \frac{dq_2}{dt} + \frac{\partial \mathbf{r}}{\partial q_3} \frac{dq_3}{dt}$$

$$= \frac{\partial \mathbf{r}}{\partial q_1} \dot{q}_1 + \frac{\partial \mathbf{r}}{\partial q_2} \dot{q}_2 + \frac{\partial \mathbf{r}}{\partial q_3} \dot{q}_3$$

$$= \sum_{i=1}^{m} \frac{\partial \mathbf{r}}{\partial q_i} \dot{q}_i \quad (2.9)$$

---

\(^{2}\) Program systems for PCs: Mathematica, Maple, MathCad, Maple is also available as Matlab-toolbox
or

\[ v(t) = \dot{r}(t) = \frac{\partial r}{\partial q} \frac{dq}{dt} \]  

(2.10)


**Jacobian-Matrix**

where \( \frac{\partial r}{\partial q} \) is the Jacobian-Matrix containing the derivatives with respect to the generalized coordinates:

\[
\frac{\partial r}{\partial q} = \begin{bmatrix}
\frac{\partial r_1}{\partial q_1} & \frac{\partial r_1}{\partial q_2} & \frac{\partial r_1}{\partial q_3} \\
\frac{\partial r_2}{\partial q_1} & \frac{\partial r_2}{\partial q_2} & \frac{\partial r_2}{\partial q_3} \\
\frac{\partial r_3}{\partial q_1} & \frac{\partial r_3}{\partial q_2} & \frac{\partial r_3}{\partial q_3}
\end{bmatrix}
\]

(2.11)

so that

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} = \begin{bmatrix}
\frac{\partial r_1}{\partial q_1} & \frac{\partial r_1}{\partial q_2} & \frac{\partial r_1}{\partial q_3} \\
\frac{\partial r_2}{\partial q_1} & \frac{\partial r_2}{\partial q_2} & \frac{\partial r_2}{\partial q_3} \\
\frac{\partial r_3}{\partial q_1} & \frac{\partial r_3}{\partial q_2} & \frac{\partial r_3}{\partial q_3}
\end{bmatrix} \begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3
\end{bmatrix}
\]

**Note:**

When we have a nested relation e.g. given by \( r(t) = r(p(q(t))) \), the procedure is almost identical. The only thing is to use the chain rule one more time to get the velocity \( v(t) = \frac{\partial r}{\partial q} \frac{dq}{dt} \).

If the position vector depends not only on the coordinates \( q(t) \) but directly on the time \( t \):

\[
r = r(q(t), t)
\]

(2.12)

then the velocity is

\[
v = \dot{r} = \frac{\partial r}{\partial t} + \frac{\partial r}{\partial q_1} \frac{dq_1}{dt} + \frac{\partial r}{\partial q_2} \frac{dq_2}{dt} + \frac{\partial r}{\partial q_3} \frac{dq_3}{dt}
\]

\[
= \frac{\partial r}{\partial q_1} \dot{q}_1 + \frac{\partial r}{\partial q_2} \dot{q}_2 + \frac{\partial r}{\partial q_3} \dot{q}_3
\]

\[
= \frac{\partial r}{\partial t} + \sum_{i=1}^{m} \frac{\partial r}{\partial q_i} \dot{q}_i
\]

(2.13)

or

\[
v(t) = \dot{r}(t) = \frac{\partial r}{\partial t} + \frac{\partial r}{\partial q} (q(t)) \dot{q}(t)
\]

(2.14)
2.1.2.2 Accelerations

To derive expressions for the accelerations we continue to differentiate the velocity in eq. (2.10) with respect to time:

$$v(t) = \dot{\mathbf{r}}(t) = \mathbf{J}_{\mathbf{r}q}(q(t)) \dot{q}(t)$$

We obtain a general relation $\mathbf{a} = \dot{\mathbf{r}}(q(t), \dot{q}(t), \ddot{q}(t))$. Using the chain rule of differentiation we get (it is a good exercise to derive this expression)

$$\mathbf{a}(t) = \ddot{\mathbf{r}}(t) = \mathbf{J}_{\mathbf{r}q} \ddot{q}(t) + \mathbf{K}_{\mathbf{r}q} \dot{q}$$  \hspace{1cm} (2.15)

Again, the Jacobian plays an important role in the first term of the right-hand side which is coupled with the second derivatives of the generalized coordinates. The second term contains quadratic expressions of the first derivatives of the $q_i$. The matrix $\mathbf{K}_{\mathbf{r}q}$ is built up by the 2nd order derivatives of the position vector with respect to the generalized coordinates:

$$\mathbf{K}_{\mathbf{r}q} = \begin{bmatrix} \frac{\partial^2 r}{\partial q_1^2} & \frac{\partial^2 r}{\partial q_1 \partial q_2} & \frac{\partial^2 r}{\partial q_1 \partial q_3} & \frac{\partial^2 r}{\partial q_2^2} & \frac{\partial^2 r}{\partial q_2 \partial q_3} & \frac{\partial^2 r}{\partial q_3^2} \end{bmatrix}$$  \hspace{1cm} (2.16)

The order corresponds to the vector $\dot{q}_Q$ of the squares of velocities

$$\dot{q}_Q = \begin{bmatrix} \dot{q}_1^2 & 2\dot{q}_1 \dot{q}_2 & 2\dot{q}_1 \dot{q}_3 & q_2^2 & 2\dot{q}_2 \dot{q}_3 & q_3^2 \end{bmatrix}^T$$  \hspace{1cm} (2.17)

The factor of 2 is due to the mixed derivatives which appear twice and which can put together in one term. All we have to do to get the accelerations is to calculate the 2nd order derivatives with respect to the $q_i$. The Jacobian is already known from the calculation of the velocities.

2.2 Kinematics of Rigid Bodies

In treating the motion of a simple particle we had to consider only translations. Now, the rotation about an arbitrary axis has to be considered, too. In the 3D space the rotation of the rigid body is given by the $\omega(t)$ vector with 3 components representing the 3 DOFs of rotation.

We investigate the motion of an arbitrary point $P$ of the rigid body. First, we have to describe the position vector and subsequently, the velocity and the acceleration vector. Knowing this for any point $P$, the state of motion is known for the whole rigid body.
We consider a fixed reference frame with origin $O$. The position vector in fig. 2.2 for a point $P$

$$r_P = r_A + r_{AP}$$  \hfill (2.18)

can be composed of the position of the reference point $A$ and the position vector pointing from $A$ to $P$. The choice of $A$ is arbitrary, too. In some applications, it makes sense to choose the center of gravity, but also a point where two rigid bodies are connected by a hinge can be chosen. The well-known Eulerian formula describes the velocity of the point $P$ of a rigid body:

$$v_P = v_A + \omega \times r_{AP}$$  \hfill (2.19)

where $\omega$ is the vector of the angular velocity (German: Winkelgeschwindigkeitsvektor). The velocity vector $v_A$ with it’s 3 components characterizes the 3 translations and the cross product deliver the 3 components of the rotation which in sum correspond to 6 DOFs of the rigid body in the 3D-space. Further differentiation with respect to time $t$ lead to the acceleration of point $P$:

$$a_P = \ddot{v}_A + \dot{\omega} \times r_{AP} + \omega \times \dot{r}_{AP}$$

$$= a_A + \dot{\omega} \times r_{AP} + \omega \times (\omega \times r_{AP})$$  \hfill (2.20)

In the plane (the motion takes place in the $x$-$y$-plane) the equations can be simplified:

$$v_P = v_A + \omega (e_z \times r_{AP})$$  \hfill (2.21)

$$a_P = a_A + \dot{\omega} (e_z \times r_{AP}) - \omega^2 r_{AP}$$  \hfill (2.22)

Now, the vector of the angular velocity $\omega$ is perpendicular to the $x$-$y$-plane and points in $z$-direction (or in negative $z$-direction). Introducing a reference frame
which is fixed on the body (and rotates with the body), and where the unit vector $\vec{e}_r$ always points in the direction of $\overline{r}_{AP}$ we get a very representation of the velocity and the acceleration:

\[
\overline{r}_P = \overline{r}_A + r \vec{e}_r \\
\overline{v}_P = \overline{v}_A + r \omega \vec{e}_\phi \\
\overline{a}_P = \overline{a}_A + r \dot{\omega} \vec{e}_\phi - r \omega^2 \vec{e}_r
\]

with

\[
r = |\overline{r}_{AP}|
\]

In the next section, we want to consider the kinematics of a rigid body as follows:

The motion of a point $P$ is described by means of the motion of a reference point $A$ in a fixed reference frame, or a so-called inertial ($I$) system (in German “Inertialsystem” ($I$)). As seen earlier, we can write again:

\[
i\overline{r}_P = i\overline{r}_A + i\overline{r}_{AP}
\]

The subscript $I$ indicates that we consider the vectors in the fixed reference frame (inertial system). It turns out that it is very useful to describe the motion of point $A$ in the ($I$)-system while we describe the motion of $P$ relative to $A$ in a the body-fixed coordinate system ($K$). ($K$) stands for the german word “körperfest” which means body-fixed). The reason for this is that when we deal with rigid bodies the distance and orientations relative to the reference point $A$ and another arbitrary point $P$ always remains constant. Hence, we express the vector $i\overline{r}_{AP}$ in the body-fixed reference frame ($K$) by $K\overline{r}_P'$, where we must keep in mind that the ($K$)-frame has it’s origin in point $A$. Furthermore, we must consider that the instantaneous orientation of the ($K$)-frame in general is not identical with the fixed frame ($I$). Due to the 3D-motion of the body, the ($K$)-frame is rotated against the fixed ($I$)-frame.
Figure 2.4: Different reference frames: fixed ref. frame (I) and moving frame (K) fixed on the moving body

We express this rotation by a rotation matrix $A_{IK}$ which has dimension $3 \times 3$. We can also say: $A_{IK}$ represents the transformation of a vector from the (K)- into (I)-coordinates:

$$I^r_P = I^r_A + A_{IK}K^r_P$$ (2.28)

Matrix $A_{IK}$ can characterized by 3 rotation parameters, e.g. 3 angles. We shall see later how the rotation matrix is built up.

### 2.2.1 Transformation Matrices

For the determination of the transformation matrix $A_{IK}$ by three rotational angles defined by three elementary rotations about the $x-$axis with angle $\alpha$, about $y-$axis with angle $\beta$ and about $z-$axis with angle $\gamma$. The elementary rotation matrix $A_{\alpha}$ maps the vector $I^r_P$ in the fixed reference frame to the vector $K^r_P$. The (K)-frame is rotated about the $x-$axis and the angle of rotation is $\alpha$.

$$
A_{\alpha} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha \\
0 & -\sin \alpha & \cos \alpha
\end{bmatrix}
$$ (2.29)
In the same way the rotations about the $y$ and the $z$ axes are carried out. The rotation matrices which correspond to these rotations are denoted by $A_\beta$ and $A_\gamma$, respectively:

\[
A_\beta = \begin{bmatrix}
\cos \beta & 0 & -\sin \beta \\
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta
\end{bmatrix}
\]  
(2.30)

\[
A_\gamma = \begin{bmatrix}
\cos \gamma & \sin \gamma & 0 \\
-\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  
(2.31)

Now, we perform all three rotations subsequently. The first rotation about the $x$ axis yields:

\[
K^{(1)} = A_\alpha I_L
\]  
(2.32)

after that we carry out the rotation about the $y$ axis

\[
K^{(2)} = A_\beta K^{(1)} = A_\beta A_\alpha I_L
\]  
(2.33)

and after the third rotation we get the final position:

\[
K^{(3)} = A_\gamma A_\beta A_\alpha I_L
\]  
(2.34)

Leaving the superscript away we get:

\[
K_L = A_\gamma A_\beta A_\alpha I_L
\]  
(2.35)

We can see that the transformation matrix is a product of elementary rotations. From matrix calculus we know that we are not allowed to change the order of multiplication of the single matrices. This leads to a different matrix product. Physically, this means that we are not allowed to change the order of rotation. This would lead to a different final position.

\[
A_{KI} = A_\gamma A_\beta A_\alpha
\]  
(2.36)

For our application we also need the inverse mapping $A_{IK}$. We simply have to take the inverse of the matrix $A_{KI}$:

\[
A_{IK} = A_{KI}^{-1} = (A_\gamma A_\beta A_\alpha)^{-1} = A_\alpha^{-1} A_\beta^{-1} A_\gamma^{-1}
\]  
(2.37)
Because the transformation matrices are orthogonal, we can replace the inverse by the much simpler transpose of the matrix

\[ A_{IK}^{-1} = A_{IK}^T \]

so that

\[ I_K = A_{IK} K_L \]

This final matrix \( A_{IK} \) is valid only for the pre-defined order of rotation about the pre-defined order of rotation about \( x-, y-, z-\) axes. The order of rotation can be arbitrary chosen, in such cases, the transformation matrix \( A_{IK} \) is to be recalculated for each chosen order. Two different orders of rotation are mostly used in the machine dynamic studies and are called Cardinian and Euler angles.

### 2.2.1.1 Cardanian Angles

The so-called Cardanian angles represent one possibility to describe the rotation of a rigid body in a unique way. What we are doing is to carry out the 3D rotation by 3 subsequent rotations about the 3 axes of the body-fixed reference frame. It is important to note, that is not allowed to change the order of the three rotations! A change of the order of the subsequent rotation usually will lead to a different position at the end of the rotations. 

The Cardanian angles are defined in a way to carry-out the three subsequent rotations about the \( x-y-z \) axes.

\[ A_{IK}^{Cardanian} = (A_\alpha A_\beta A_\gamma)^T \] (2.41)

---

3 Finite angles of rotation do not have vector character, they do not obey the commutative law. The order of the rotations is absolutely important. Contrary to this, infinitesimal small rotations and angular velocities (which are based on infinitesimal small angles) have all properties of a vector.
2.2 Kinematics of rigid bodies

Elementary rotations

\[
A_\alpha = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha \\
0 & -\sin \alpha & \cos \alpha
\end{bmatrix}
\]

\[
A_\beta = \begin{bmatrix}
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta \\
\cos \gamma & \sin \gamma & 0
\end{bmatrix}
\]

\[
A_\gamma = \begin{bmatrix}
-\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Rotation matrix

\[
A_{IK} = \begin{bmatrix}
\cos \beta \cos \gamma & -\cos \beta \sin \gamma & \sin \beta \\
\cos \alpha \sin \gamma + \sin \alpha \sin \beta \cos \gamma & \cos \alpha \cos \gamma - \sin \alpha \sin \beta \sin \gamma & -\sin \alpha \cos \beta \\
\sin \alpha \sin \gamma - \cos \alpha \sin \beta \cos \gamma & \sin \alpha \cos \gamma + \cos \alpha \sin \beta \sin \gamma & \cos \alpha \cos \beta
\end{bmatrix}
\]

Angular velocities

\[
I \omega = \begin{bmatrix}
1 & 0 & \sin \beta \\
0 & \cos \alpha & -\sin \alpha \cos \beta \\
0 & \sin \alpha & \cos \alpha \cos \beta
\end{bmatrix}
\]

\[
\dot{\alpha} = \begin{bmatrix}
\dot{\alpha} \\
\dot{\beta} \\
\dot{\gamma}
\end{bmatrix}
\]

\[
K \omega = \begin{bmatrix}
\cos \beta \cos \gamma & \sin \gamma & 0 \\
-\cos \beta \sin \gamma & \cos \gamma & 0 \\
\sin \beta & 0 & 1
\end{bmatrix}
\]

Kinematic equation

\[
\begin{bmatrix}
\dot{\alpha} \\
\dot{\beta} \\
\dot{\gamma}
\end{bmatrix} = \frac{1}{\cos \beta}
\begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma \cos \beta & \cos \gamma \cos \beta & 0 \\
-\sin \beta \cos \gamma & \sin \gamma \sin \beta & 1
\end{bmatrix}
\begin{bmatrix}
K \omega_x \\
K \omega_y \\
K \omega_z
\end{bmatrix}
\]

\[
= \frac{1}{\cos \beta}
\begin{bmatrix}
0 & \cos \beta \cos \alpha & \cos \beta \sin \alpha \\
0 & -\sin \alpha & \cos \alpha
\end{bmatrix}
\begin{bmatrix}
I \omega_x \\
I \omega_y \\
I \omega_z
\end{bmatrix}
\]

Table 1: Cardanian angles

This means: at first we rotate about the x-axis with an angle \( \alpha(t) \). This leads to a new position of the rotated coordinate system \( x'-y'-z' \), where the \( x' \)-axis and the old x-axis are still identical. After that we rotate about the new \( y' \)-axis with an angle \( \beta(t) \) leading to the new orientation \( x''-y''-z'' \) (with \( y' = y'' \)) and finally the last rotation about the \( z'' \)-axis with an angle \( \gamma(t) \) leading to the final orientation \( x'''-y'''-z''' \) of the reference frame. An alternative possibility is to use the so-called Eulerian angles. Here, the rotations are also carried out in
3 subsequent steps but the order is different: the 3 elementary rotation have the sequence "z-x-z".

2.2.1.2 Eulerian Angles

The transformation we discussed before was based on the Cardanian angles (see chapter 2.2.1.1), which are characterized by a special order of the rotations about the x-y-z axes:

\[ A_{IK} = A_{IK}^{\text{Cardan}} \]  \hspace{1cm} (2.42)

Basically, we are free to choose any combination of rotations. We mentioned already the Eulerian angles which are based on rotations about the different axes in the order z-x-z.

\[ A_{KI} = A_{KI}^{\text{Euler}} = A_{\psi} A_{\vartheta} A_{\varphi} \]  \hspace{1cm} (2.43)

where we perform the following rotations

1. Rotation about the z-axis:

\[ A_{\psi} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (2.44)

2. Rotation About x-axis:

\[ A_{\vartheta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta & \sin \vartheta \\ 0 & -\sin \vartheta & \cos \vartheta \end{bmatrix} \]  \hspace{1cm} (2.45)

3. One more rotation about the z-axis (which now of course has a different orientation in the 3D space):

\[ A_{\varphi} = \begin{bmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (2.46)

\footnote{In the English literature the expression “Eulerian angles” is used in a more general way including the Eulerian angles as described above (Type A) and the Cardanian angles (as Eulerian angles, Type B), see F.C. Moon, Applied Dynamics.}
In general we can choose the order of the rotations, but once we have selected one, we have to retain this order unchanged. As mentioned already the formalism we derived can be applied to any rotation transformations $A_{IK}$. By this, it is possible to match the transformation to a special application without changing the general formula.

### 2.2.1.3 Small Rotations

If the angles $\alpha$, $\beta$ and $\gamma$ are small, we can simplify the trigonometric functions in the rotation matrix. With the approximation for small angles

$$\cos \alpha \approx 1 \quad (2.47)$$

and

$$\sin \alpha \approx \alpha. \quad (2.48)$$

The elementary rotations become

$$A_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & -\alpha & 1 \end{bmatrix}, \quad A_\beta = \begin{bmatrix} 1 & 0 & -\beta \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}, \quad A_\gamma = \begin{bmatrix} 1 & \gamma & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.49)$$

leading to the rotation matrix eq. (2.40)

$$A_{IK} = A_{IK} = A_{\gamma} A_{\beta} A_{\alpha} = \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix}$$

$$A_{IK} = A_{IK}^T = (A_{\gamma} A_{\beta} A_{\alpha})^T = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix}$$

where we have also neglected products of small angles like e.g. $\alpha \beta \approx 0$ etc.

### 2.2.2 Velocity in the Inertial System

The velocity of point $P$ in the inertial system can be obtained by differentiation of

$$\mathbf{V}_P = \mathbf{V}_A + A_{IK} \mathbf{V}'_P \quad (2.50)$$
with respect to time so that

\[ I \dot{r}_P = I \dot{r}_A + \dot{A}_{IK} K\dot{r}_P \]  (2.51)

Now, we can use the fact that the derivative of \( K\dot{r}_P \) with respect to time is zero, because this vector is constant in the fixed-body reference frame. And because \( K\dot{r}_P = A_{KI} I\dot{r}_P \) for any position vector, especially for

\[ K\dot{r}_P = A_{KI} I\dot{r}_P \]  (2.52)

we get velocity in the inertial system

\[ I v_P = I \dot{r}_P = I \dot{r}_A + \dot{A}_{IK} A_{KI} I\dot{r}_P \]  (2.53)

This turns out to be a skew-symmetric important formula allows to determine the velocity based on the rotation matrix. The general formulation is independent of the special rotation parameters (which were Cardanian angles in this case).

The matrix product \( \dot{A}_{IK} A_{KI} \) matrix. It contains the components of the vector of the angular velocity (see also the next section).

\[ I \tilde{\omega}_{KI} = \dot{A}_{IK} A_{KI} \]  (2.54)

Formal replacement in eq. (2.53) yields

\[ I \dot{r}_P = I \dot{r}_A + I \tilde{\omega}_{KI} I\dot{r}_P \]  (2.55)

or

\[ I v_P = I v_A + I \tilde{\omega}_{KI} I\dot{r}_P \]  (2.56)

where

\[ I \tilde{\omega}_{KI} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \]  (2.57)

The corresponding vector of the angular velocity is

\[ I \tilde{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \]  (2.58)

This allows us to calculate the three components of the angular velocities from the product \( I \tilde{\omega}_{KI} = \dot{A}_{IK} A_{KI} \).
### 2.2 Kinematics of Rigid Bodies

#### 2.2.3 Relation Between Matrix and Vector Representation of the Velocity

Comparing the relation

\[
Iv_P = Iv_A + \tilde{\omega}_K Ii_P
\]

with the *Eulerian* equation discussed in section 2.2:

\[
v_P = v_A + \omega \times i_{AP}
\]

we can see immediately the similarity of both representations of the velocity. The matrix representation is very useful as we could see when we derive the angular velocity from the rotation matrix (in this case based on Cardanian angles).

**Note:**

A vector product \(a \times b\) of two vectors \(a\) and \(b\) can also be written as a matrix-vector-multiplication

\[
a \times b = \tilde{a} b
\]

with

\[
a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad \text{und} \quad b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}
\]

**Note:** Proof of skew-symmetry

We start with the relation that forward and subsequent backward transformation leads to the identity, where \(I\) is the identity matrix:

\[
\overset{\rightarrow}{A}_{IK} \overset{\rightarrow}{A}_{KI} = I
\]

Differentiation with respect to time using the product rule leads to

\[
\frac{d}{dt} (\overset{\rightarrow}{A}_{IK} \overset{\rightarrow}{A}_{KI}) = \dot{\overset{\rightarrow}{A}}_{IK} \overset{\rightarrow}{A}_{KI} + \overset{\rightarrow}{A}_{IK} \dot{\overset{\rightarrow}{A}}_{KI} = 0
\]

The derivative of the constant identity matrix is zero. This gives

\[
\dot{\overset{\rightarrow}{A}}_{IK} \overset{\rightarrow}{A}_{KI} = -\overset{\rightarrow}{A}_{IK} \dot{\overset{\rightarrow}{A}}_{KI}
\]

Making use of the orthogonality property of the transformation matrices \(\overset{\rightarrow}{A}_{KI} = \overset{\rightarrow}{A}_{IK}^{-1} = \overset{\rightarrow}{A}_{IK}^T\) we get

\[
\dot{\overset{\rightarrow}{A}}_{IK} \overset{\rightarrow}{A}_{KI} = -\overset{\rightarrow}{A}_{IK} \dot{\overset{\rightarrow}{A}}_{KI} = -\overset{\rightarrow}{A}_{KI}^T \dot{\overset{\rightarrow}{A}}_{IK} = -\left(\dot{\overset{\rightarrow}{A}}_{IK} \overset{\rightarrow}{A}_{KI}\right)^T
\]

which shows us the desired skew-symmetry.
We can show easily that the components of vectors \( \mathbf{a} \) have to be arranged in the matrix scheme as follows:

\[
\mathbf{\tilde{a}} = \begin{bmatrix}
0 & -a_z & a_y \\
a_z & 0 & -a_x \\
-a_y & a_x & 0 \\
\end{bmatrix}
\]  
(2.63)

This matrix is skew-symmetric. The symbol “\( \sim \)” indicates that the vector components of a vector \( \mathbf{a} \) have to be arranged according to the scheme of eq. (2.63). We can see that

\[
\mathbf{a} \times \mathbf{b} = \mathbf{\tilde{a}} \mathbf{b} = \begin{bmatrix}
0 & -a_z & a_y \\
a_z & 0 & -a_x \\
-a_y & a_x & 0 \\
\end{bmatrix}
\begin{bmatrix}
b_x \\
b_y \\
b_z \\
\end{bmatrix} = \begin{bmatrix}
a_y b_z - a_z b_y \\
a_z b_x - a_x b_z \\
a_x b_y - a_y b_x \\
\end{bmatrix}.
\]  
(2.64)

We know that \( \mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}) \) so that we can derive that

\[
\mathbf{\tilde{a}} \mathbf{b} = -\mathbf{\tilde{b}} \mathbf{a}.
\]  
(2.66)

### 2.2.4 Velocity in the Body-Fixed Reference Frame

The velocity expressed in inertial system was:

\[
\mathbf{v}_P = \mathbf{v}_A + \mathbf{\tilde{\omega}}_K \mathbf{r}_P
\]

We know that we can transform a vector from the \( I \) into the \( K \)-system by means of the rotation matrix \( \mathbf{A}_{KI} \) and vice versa. This is valid also for the velocity vector:

\[
\mathbf{v}_P = \mathbf{A}_{KI} \mathbf{v}_A \]  
(2.56)

\[
= \mathbf{A}_{KI} \mathbf{v}_A + \mathbf{A}_{KI} \mathbf{\tilde{\omega}}_K \mathbf{r}_P
\]  
(2.57)

or

\[
\mathbf{v}_P = \mathbf{Kv}_A + \mathbf{\tilde{\omega}}_K \mathbf{r}_P
\]  
(2.68)

In analogy to the considerations of chapter. 2.2.2 we can derive the angular velocity, but now expressed by the components of the \( K \)-system

\[
\mathbf{\tilde{\omega}}_{IK} = \mathbf{\dot{A}}_{KI} \mathbf{A}_{IK}
\]  
(2.71)
Performing some elementary steps
\[
K\tilde{\omega}_{IK} = A_{KI}A_{IK}
\]
\[
= A_{KI}A_{IK} (A_{KI}A_{IK})
\]
\[
= A_{KI} (A_{IK} A_{KI}) A_{IK}
\]
\[
= A_{KI} (i\tilde{\omega}_{KI}) A_{IK}
\]
\[
= A_{IK}^T (i\tilde{\omega}_{KI}) A_{IK}
\]
we get the well-known transformation law
\[
K\tilde{\omega}_{IK} = A_{IK}^T (i\tilde{\omega}_{KI}) A_{IK}.
\]
(2.72)
This allows us to express the velocity in terms of the coordinates of the body fixed coordinate system:
\[
K\nu_P = K\nu_A + K\tilde{\omega}_{IK} K\nu_P
\]
(2.73)

**Note:**
We want to note that the velocity of point \(A\) is not identical to the time derivative of \(K\nu_A\): \(K\nu_A \neq K\dot{\nu}_A\).

However we obtain the absolute velocity of \(A\) in components of the \(K\)-reference frame:
\[
K\nu_A = A_{KI} I\nu_A = A_{KI} I\dot{\nu}_A.
\]
(2.74)
The absolute velocity expressed in the moving, body-fixed reference frame can be obtained only by differentiating the position vector in the inertial reference frame \(I\). Only the time derivative with respect to an inertial system yields an absolute velocity (otherwise it is only a relative velocity). So, the procedure is:

- first calculate the velocity in the \(I\)-System,
- then transform it to the \(K\)-System using the rotation matrix.

### 2.2.5 Accelerations in the Inertial System

In order to derive an expression for the accelerations we have to differentiate once more with respect to time \(t\):
\[
i\ddot{\nu}_P = i\ddot{\nu}_A + A_{IK} K\nu'_P
\]
(2.75)
with

\[ K \dot{\ell}_P = A_{KI} \dot{\ell}_P \]  

(2.76)

we get

\[ \dot{\ell}_P = \dot{\ell}_A + A_{KI} \dot{A}_{KI} \dot{\ell}_P . \]  

(2.77)

With these relations we already can calculate the accelerations.

We investigate the product of the transformation matrix

\[ \frac{d}{dt} \left( A_{KI} A_{KI} \right) = \dot{A}_{KI} A_{KI} + A_{KI} \dot{A}_{KI} = \dot{A}_{KI} A_{KI} + A_{KI} \left( A_{KI} A_{IK} \right) A_{KI} \]  

(2.78)

(2.79)

The product \( \dot{A}_{KI} A_{KI} \) is nothing else but the matrix of the angular velocities \( \dot{\omega}_{KI} \), so that it follows that

\[ \dot{A}_{KI} A_{KI} = \dot{\omega}_{KI} + \left( \dot{\omega}_{KI} \right) A_{KI} \]  

(2.80)

This yields the absolute accelerations

\[ \dot{\ell}_P = \dot{\ell}_A + \left( \dot{\omega}_{KI} + \dot{\omega}_{KI} \dot{\omega}_{KI} \right) \dot{\ell}_P \]  

(2.81)

or shorter (indices)

\[ \dot{\ell}_P = \dot{\ell}_A + \left( \dot{\omega} + \dot{\omega} \dot{\omega} \right)_{KI} \dot{\ell}_P . \]  

(2.82)

or

\[ \dot{a}_P = \dot{a}_A + \left( \dot{\omega} + \dot{\omega} \dot{\omega} \right)_{KI} \dot{\ell}_P . \]  

(2.83)

We compare this result to the vector formula which was presented earlier eq. (2.20):

\[ \dot{a}_P = \dot{a}_A + \dot{\omega} \times \ell_{AP} + \omega \times (\dot{\omega} \times \ell_{AP}) \]

and we can see that there is a perfect analogy.

2.2.6 Acceleration in the Body-Fixed Reference Frame

The transformation into the \( K \)-frame is identical as with the velocities. We multiply by \( A_{KI} \):

\[ A_{KI} \dot{a}_P = A_{KI} \dot{a}_A + A_{KI} \left( \dot{\omega} + \dot{\omega} \dot{\omega} \right)_{KI} \dot{\ell}_P \]  

(2.84)

\[ = A_{KI} \dot{a}_A + A_{KI} \left( \dot{\omega} + \dot{\omega} \dot{\omega} \right)_{KI} A_{KI} \dot{\ell}_P \]  

(2.85)
The acceleration of the reference point $A$ is

$$K \ddot{a}_A = \dot{A}_{KI} \ddot{r}_A = \dot{A}_{KI} \ddot{a}_A \tag{2.87}$$

### 2.2.7 Angular Accelerations

To get a transformation rule for the vector of the angular accelerations in the $I$ and the $K$-frame, respectively, we start with:

$$I \dot{\omega} = \frac{d}{dt} (I \omega) \tag{2.88}$$

The transformation leads to

$$A_{KI} I \dot{\omega} = A_{KI} \frac{d}{dt} (I \omega)$$

$$= A_{KI} \frac{d}{dt} (A_{IK} K \omega)$$

$$= A_{KI} (A_{IK} K \omega + A_{IK} K \dot{\omega})$$

$$= A_{KI} A_{IK} K \omega + A_{KI} A_{IK} K \dot{\omega}$$

$$= K \ddot{\omega}_{IK} K \omega + K \dot{\omega}$$

Because the cross product of two identical vectors is zero:

$$K \ddot{\omega}_{IK} K \omega = K \omega \times K \omega = 0 \tag{2.89}$$

we finally obtain

$$K \ddot{\omega} = A_{KI} I \dot{\omega} \tag{2.90}$$

### 2.2.8 Systems with Constraints

In section 1.2.2 we already discussed the reduction of the number of DOF by constraints. The number of DOFs for a motion in the 3D-space for $n$ rigid bodies was given by

$$f = 6n - c \tag{2.91}$$

where $c$ denotes the number of constraints. For a set of $n$ particles (no rotations) we only get

$$f = 3n - c \tag{2.92}$$
The position of a particle can be described by the position vector \( r \). For \( n \) particles, we have \( n \) position vectors \( r_i \)

\[
\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, K, q, f) \quad i = 1, 2, K, n
\]  

(2.93)

The description using the position vectors of some \((n)\) representative points is known as the configuration space. However as can be seen in fig. 2.5, the description in the configuration space with \(3\) quantities \((x, y, z)\) is over-specified, the number of DOFs is only \(2\). Thus, two generalized coordinates \(q_1\) and \(q_2\) are sufficient to describe the position of the particle in a unique way.

![Figure 2.5: Example for a system with constraints: the particle can only move in the \(q_1-q_2\)-plane](image)

### 2.2.8.1 Holonomic Constraints

The constraints are formulated mathematically as an vector equation:

\[
\Phi(y) = 0
\]  

(2.94)

where \(y\) represents all the coordinates necessary to describe the constraint. The vector \(\Phi\) has as many components as mathematical constraints are available. If time \(t\) does not appear explicitly, the constraint is said to be scleronomic.

If the time appears explicitly

\[
\Phi(y, t) = 0
\]  

(2.95)

we call the constraint rheonomic.

Examples:
1. A particle which can only move along a parabolic curve:

Parabola: \[ y = x^2 \] leads to \[ \Phi(x, y) = y - x^2 = 0 \] (2.96)

This constraint is *holonomic-scleronomic* because the time does not appear explicitly.

2. A particle is fixed with a thread of constant length \( L \). The other end of the thread is fixed to origin of the coordinate system. The particle is constrained to move on a spherical surface with radius \( L \):

sphere: \[ x^2 + y^2 + z^2 = L^2 \] or \[ \Phi(x, y, z) = x^2 + y^2 + z^2 - L^2 = 0 \] (2.97)

The particle has 2 DOFs. The constraint is also *holonomic-scleronomic*. The description using the coordinates \( x, y, z \) is over-specified, only two quantities e.g. the two angles of spherical coordinates are required (2 generalized coordinates). If the length of the thread is depending on time: \( L = L(t) \), the constraint becomes *holonomic-rheonomic*, the constraint is now

\[ \Phi(x, y, z, t) = x^2 + y^2 + z^2 - L(t)^2 = 0 \text{ e.g.} \ L(t) = L_0 + \Delta L \cos(\omega t) \] (2.98)

In spherical coordinates the motion of the particle can be described in a simple way because only the radial component \( r \) is influenced by the constraint:

\[ \Phi = r - L(t) = 0 \] (2.99)

so that

\[ \mathbf{x} = \begin{bmatrix} L(t) \cos \psi \sin \vartheta \\ L(t) \sin \psi \sin \vartheta \\ L(t) \cos \vartheta \end{bmatrix} = \mathbf{r}(q_1, q_2, t) \] (2.100)

If the velocities and accelerations have to be calculated, we proceed as described in chapter 2.1.2.

For *holonomic-scleronomic* constraints the velocities can be calculated by means of the *Jacobian*

\[ \dot{x} = \sum_{i=1}^{f} \frac{\partial r}{\partial q_i} \dot{q}_i = \mathbf{J}_{rq} \dot{q} \] (2.101)

For *holonomic-rheonomic* constraints time \( t \) appears explicitly in the constraint equations so that the direct derivative with respect to \( t \) has to be considered:

\[ \ddot{x} = \frac{\partial r}{\partial t} + \sum_{i=1}^{f} \frac{\partial r}{\partial q_i} \ddot{q}_i = \mathbf{J}_{rq} \ddot{q} \] (2.102)

The calculation of the accelerations also follows chapter 2.1.2.
2.2.8.2 Non-holonomic Constraints

While holonomic constraints only deal with the coupling of geometric quantities, non-holonomic constraints couple also velocities. Hence, the implicit mathematical description of a non-holonomic constraints has the form:

$$\Psi = \Psi(y, \dot{y}, t) = 0 \quad (2.103)$$

In some special cases the non-holonomic constraints can be integrated in order to obtain equivalent geometric constraints, but in general non-holonomic constraints are not integrable. This means that the position coordinates cannot be derived from the integration of the constraint equations. This means that we can get a certain position on different paths.

As an example for a non-holonomic system we consider a rolling coin or wheel which moves along a curved path in the $x$-$y$-plane without sliding. For a motion with pure rolling the velocity of the center point of the wheel is coupled with the rotational speed.

![Figure 2.6: A rolling wheel as an example for a non-holonomic system](image)

The wheel as a free rigid body has 6 DOFs.

$$y = (x, y, z, \alpha, \beta, \gamma)^T \quad (2.104)$$

The center of the wheel has a constant height described by the holonomic constraint $z = R$ (where $R$ is the radius of the wheel).

$$\Phi_1 = z - R = 0, \quad (2.105)$$

If we further assume that the wheel is always in a vertical position, a second holonomic constraint can be added:

$$\Phi_2 = \alpha - \alpha_0 \quad \text{with} \quad \alpha_0 = 0 \quad (2.106)$$

The state of the wheel can be described by 4 DOFs (4 generalized coordinates, $f = 4$)

$$q = (x, y, \beta, \gamma)^T \quad (2.107)$$
where the angle $\gamma$ describes the tangent of the path and hence the actual direction of the rolling wheel in the $x$-$y$-plane, the angle $\beta$ describes the rotation of the wheel along the path, see fig. 2.7.

![Figure 2.7: Rolling wheel](image)

The relation between the two angles $\beta$ and $\gamma$ and the translation of the center of the wheel given by the coordinates $(x, y, z = R)$ can be derived by geometric considerations when we rotate the wheel about an infinitesimal small angle $d\beta$. With $ds = -Rd\beta$ we get

\begin{align*}
  dx &= ds \cos \gamma = -Rd\beta \cos \gamma \quad (2.108) \\
  dy &= ds \sin \gamma = -Rd\beta \sin \gamma \quad (2.109)
\end{align*}

If we relate the infinitesimal small displacements $dx$ and $dy$ to an infinitesimal small time interval $dt$ we get the velocities

\begin{align*}
  \dot{x} &= \frac{dx}{dt} = -R \frac{d\beta}{dt} \cos \gamma = -R \dot{\beta} \cos \gamma \quad (2.110) \\
  \dot{y} &= \frac{dy}{dt} = -R \frac{d\beta}{dt} \sin \gamma = -R \dot{\beta} \sin \gamma \quad (2.111)
\end{align*}

Now we have found the coupling between the different velocities: the velocities \( \dot{x} \) and \( \dot{y} \) are determined by the non-holonomic constraints:

\begin{align*}
  \Psi_1(x, \dot{\beta}, \gamma) = \dot{x} + R \dot{\beta} \cos \gamma &= 0 \quad (2.112) \\
  \Psi_2(y, \dot{\beta}, \gamma) = \dot{y} + R \dot{\beta} \sin \gamma &= 0 \quad (2.113)
\end{align*}

In the special case that the angle $\gamma = \gamma_0 = const.$, (that means the path is a straight line) the constraint can be integrated:

\begin{align*}
  x - x_0 &= -R \cos \gamma_0 (\beta - \beta_0) \quad (2.114) \\
  y - y_0 &= -R \sin \gamma_0 (\beta - \beta_0) \quad (2.115)
\end{align*}
Now, we have a unique relation between the angle $\beta$ and the two coordinates $x, y$. This means that we have only 1 DOFs: the motion can be fully described by the angle $\beta$. Presetting the angle $\gamma = \gamma_0 = \text{const.}$ changes the problem from a non-holonomic to a holonomic system.

In technical systems holonomic constraints occur more frequently than non-holonomic ones.

### 2.3 Relative Motion of a Particle

As is well-known, Newton’s law is only valid in an inertial system. But often it is more convenient to describe the motion of a particle in a moving reference frame. This is the case when the interest lies in examining the motion of particles relative to spinning bodies or the motion of spinning bodies relative to a fixed reference frame. (Examples for this are the motion of a body on the rotating earth or the motion of gyroscopes or similar devices with respect to the inertial space). Other examples are friction where we must know the relative velocity between the two machine parts which are in contact, or the motion of car occupants in an accelerated car during a car crash. We have to distinguish between the coordinate frame in which the vector components are written and the coordinate frame in which the time derivatives are taken.

We consider two reference frames: one is the inertial system $(I)$ which is fixed and the second is a moving reference frame $(R)$. We wish to describe the motion of the particle $P$ with respect to the $(R)$-frame. The reference frame $(R)$ can perform a translation and a rotation.
2.3.1 Relation Between Absolute and Relative Velocity

If we look at point $P$ from the origin of the inertial system $(I)$ we can describe it by the position vector

$$\mathbf{r}_P^I = \mathbf{r}_0^I + \mathbf{r}_{0P}$$  

(2.117)

Now we replace $\mathbf{r}_P^I$ (as is done in the chapter on kinematics of rigid bodies) by the corresponding vector in the moving reference frame $(R)$ and the rotation matrix:

$$\mathbf{r}_P^I = \mathbf{r}_0^I + A_{IR} R \mathbf{r}_{0P}$$  

(2.118)

Contrary to the case where we considered a point $P$ on the rigid body which had always a fixed distance from the origin of the moving frame, now we have to consider that also the position $\mathbf{r}_P^I$ of point $P$ relative to frame $(R)$ may change. Hence, we have to look at the temporal change in the reference frame $(R)$ which we denote by the symbol $\frac{d'}{dt}$ which is the derivative with respect to time but in the moving frame, while the derivative $\frac{d}{dt}$ is the absolute change with respect to the fixed system.

The absolute velocity now is:

$$\mathbf{v}_P^I = \dot{\mathbf{r}}_P^I = \dot{\mathbf{r}}_0^I + \dot{\mathbf{r}}_{0P} = \dot{\mathbf{r}}_0^I + A_{IR} \frac{d'}{dt} (R \mathbf{r}_{0P})$$  

(2.119)

with the relative velocity:

$$R \mathbf{v}_{rel}^I = \frac{d'}{dt} (R \mathbf{r}_{0P})$$  

(2.120)
We use the rotation matrix $A_{RI}$ to transform the vector in $(I)$-components to the $(R)$ system.

$$R^IP = A_{RI} I^IP$$

$$= A_{RI} I^0 + A_{RI} \dot{A}_{IR} R^0P + A_{RI} \dot{A}_{IR} R^0 \omega_{rel}$$

$$R^IP = R^0P + A_{RI} \dot{A}_{IR} R^0P + R^0 \omega_{rel}$$

We already know the expression from earlier considerations $A_{RI} \dot{A}_{IR}$ as the skew-symmetric matrix of the angular velocities:

$$R^0 \omega_{IR} = A_{RI} \dot{A}_{IR}$$

so that the velocity is

$$R^0 = R^0P + R^0 \omega_{IR} R^0P + R^0 \omega_{rel}$$

This expression for the absolute velocity has three different terms which we can explain intuitively: The last term is the relative velocity of $P$ with respect to the moving reference frame $(R)$; the first two terms represent the velocity of point $P$ when we consider this point to be fixed in the $(R)$-frame. They are the velocity of the origin of $(R)$-frame and the rotation of the $(R)$-frame. We call these two terms guidance velocity $v_g$ (German: “Führungsgeschwindigkeit”).

With the guidance velocity we can write

$$R^0 = R^0P + R^0 \omega_{rel}.$$
A person sitting at the origin of the moving \((R)\)-frame only sees the relative velocity, while a person sitting in the inertial system sees all guidance velocity plus relative velocity.

Using the rotation matrix it is very easy to express the velocity in coordinates of the fixed coordinate system:

\[
Iv = A_{IR}RLv \\
= A_{IR}RLv_g + A_{IR}RLv_{rel} \\
= Iv_g + Iv_{rel}
\]

\(2.127\) \(2.128\) \(2.129\)

**Note:**
In the classical notation of the relative kinematics the vector product is used which we have already shown to be equivalent to the matrix notation. In the classical notation of the relative kinematics the vector product is used which we have already shown to be equivalent to the matrix notation.

\[
Iv = \dot{v}_0 + \omega \times LOP + v_{rel} = v_g + v_{rel}
\]

\(2.130\)

### 2.3.2 Relation Between Absolute and Relative Acceleration

The acceleration can be obtained by further differentiation:

\[
Ia_P = I\ddot{r}_P = I\ddot{r}_0 + I\ddot{L}_0P
\]

\(2.131\)

or

\[
Ia_P = Ia_0 + \left(\dot{A}_{IR}RL0P + A_{IR}RL0P\right)\dot{r}_P
\]

\(2.132\)

This gives:

\[
Ia_P = Ia_0 + \dot{A}_{IR}RL0P + 2\dot{A}_{IR}RL0P + A_{IR}RL0P
\]

\(2.133\)

The derivatives with the prime indicate again that these express the changes with respect to the moving frame. With the relative velocity of the last section and the relative acceleration

\[
R\ddot{u}_{rel} = R\dddot{L}_{0P}
\]

\(2.134\)

we can write

\[
Ia_P = Ia_0 + \dot{A}_{IR}RL0P + 2\dot{A}_{IR}RL0P + A_{IR}Ra_{rel}
\]

\(2.135\)

Transformation as we have done it with the velocities now yields:

\[
R\ddot{u}_P = A_{RI}Ia_P = A_{RI}Ia_0 + A_{RI}A_{IR}RL0P \ldots \\
\dot{L}_{rel} + 2A_{RI}A_{IR}L_{rel} + A_{RI}A_{IR}Ra_{rel}
\]

\(2.136\)
\[ R^a_P = R^a_0 + A_{RI} \tilde{\omega}_{IR} R^0_0 + 2A_{RI} \tilde{\omega}_{IR} R^0_{rel} + R^0_{rel} \]  

(2.137)

The product terms \( A_{RI} \tilde{\omega}_{IR} \) and \( A_{RI} \tilde{\omega}_{IR} \) can be expressed by the matrix of the angular velocities:

\[ R \tilde{\omega}_{IR} = A_{RI} \tilde{\omega}_{IR} \]

and using eq. (2.80)

\[ A_{RI} \tilde{\omega}_{IR} = R \tilde{\omega}_{IR} + \tilde{R} \tilde{\omega}_{IR} R \tilde{\omega}_{IR} \]

we get

\[ R^a_P = R^a_0 + \tilde{R} \tilde{\omega}_{IR} R^0_0 + \tilde{R} \tilde{\omega}_{IR} R^0_{rel} + 2\tilde{R} \tilde{\omega}_{IR} R^0_{rel} + R^0_{rel} \]  

(2.138)

We can see that the absolute acceleration expressed in terms of the \( R \)-frame consists of five terms. The acceleration terms now appear to be less intuitive as the velocities.

The last one is the relative acceleration of point \( P \) in the moving coordinate system which the observer sitting in the moving \( R \)-frame can see. We get this quantity by differentiating the position vector in the moving frame twice with respect to time, regardless how the \( R \)-frame is moving.

The first three terms are the guidance acceleration (German: “Führungsbeschleunigung”):

\[ R^a_g = R^a_0 + \tilde{R} \tilde{\omega}_{IR} R^0_0 + \tilde{R} \tilde{\omega}_{IR} R^0_{rel} \]  

(2.140)

Coriolis acceleration

The term with the factor of 2 is the Coriolis acceleration

\[ R^a_{Cor} = 2R^0_{rel} = 2R^0_{rel} \tilde{\omega}_{IR} \tilde{\omega}_{IR} \]

(2.139)

which appears when a point \( P \) moves in the \( R \)-frame with a relative velocity \( \tilde{\omega}_{rel} \) in a rotating reference frame. (This happens always when we move on the surface of the earth.)

Centripetal acceleration

This expression has three terms: the acceleration of the origin of the \( R \)-frame, a term describing the part of the acceleration resulting from angular acceleration and the last one with quadratic terms of the angular velocities is the centripetal acceleration. We can write

\[ R^a = R^a_g + R^a_{Cor} + R^a_{rel} \]  

(2.141)
If we are interested to get the acceleration in components of the inertial system all have to do is multiplying by the rotation matrix:

$$\mathbf{a} = \mathbf{A}_IR\mathbf{a}$$  \hspace{1cm} (2.142)

**Note:**
In classical notation using the cross product the acceleration have the form:

$$\mathbf{a}_g = \mathbf{a}_0 + \dot{\mathbf{\omega}} \times \mathbf{r}_{DP} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}_{DP})$$  \hspace{1cm} (2.143)

$$\mathbf{a}_{Cor} = 2\mathbf{\omega} \times \mathbf{v}_{rel}$$  \hspace{1cm} (2.144)

$$\mathbf{a}_{rel} = \frac{d^2 \mathbf{r}_{DP}}{dt^2}$$  \hspace{1cm} (2.145)

where we can recognize the corresponding terms of the matrix notation.

### 2.3.3 Summary of the Formula for Relative Kinematics

**Summary**

1. Absolute velocity in moving frame

$$Rv_{P} = Rv_{0} + R\ddot{\mathbf{\omega}}_IRRL_{DP} + R\dot{L}_{DP}$$  \hspace{1cm} (2.146)

with

$$Rv_{g} = Rv_{0} + R\ddot{\mathbf{\omega}}_IRRL_{DP}$$  \hspace{1cm} guidance velocity  \hspace{1cm} (2.147)

$$Rv_{rel} = R\dot{L}_{DP}$$  \hspace{1cm} relative velocity  \hspace{1cm} (2.148)

The observer in the moving system gets aware of the relative velocity without knowing anything about the motion of the reference system.

- Classical notation with cross product:

$$v_{P} = v_{0} + \dot{\mathbf{\omega}} \times \mathbf{r}_{DP} + \frac{d'\mathbf{L}_{DP}}{dt}$$  \hspace{1cm} (2.149)

Derivation $\frac{d'}{dt}$ means derivation with respect to time in the moving system.

2. Absolute acceleration expressed in the $R$-system:

$$Ra_{P} = Ra_{0} + R\ddot{\mathbf{\omega}}_IRRL_{DP} + R\ddot{\mathbf{\omega}}_IRRL_{DP} + \ldots$$  \hspace{1cm} (2.150)
with

\[ R \ddot{a}_g = R \ddot{a}_0 + R \ddot{\omega}_I R \dddot{\mathbf{r}}_0 P + \ldots \quad \text{Relative acceleration} \]
\[ \ldots + R \dddot{\omega}_I R \dddot{\omega}_I R \dddot{\mathbf{r}}_0 P \]
\[ R \ddot{a}_{Cor} = 2 R \dddot{\omega}_I R \dddot{\mathbf{r}}_0 P = 2 R \dddot{\omega}_I R \dddot{v}_{rel} \quad \text{Coriolis-acceleration} \]
\[ R \ddot{a}_{rel} = R \dddot{\mathbf{r}}_0 P \quad \text{Guidance acceleration} \]

- Classical notation

\[
\dddot{a}_P = \dddot{a}_0 + \dddot{\omega} \times \dddot{\mathbf{r}}_0 P + \dddot{\omega} \times (\dddot{\omega} \times \dddot{\mathbf{r}}_0 P) + 2 \frac{\dddot{\omega} \times \dddot{v}_{rel}}{2 \dddot{v}_{rel}} + \frac{d^2 \dddot{L}_0 P}{dt^2} \]

(2.151)

Derivation \( \frac{d^2}{dt^2} \) means derivation with respect to time in the moving system.
Here, we investigate the interaction of motion of a body or system of bodies and the forces and moments causing this motion.

3.1 KINETICS OF A SINGLE PARTICLE

The idealization of a “real world body” as a particle is admissible, if only the translation of the body is of interest. The mass of the body is concentrated to the centre of gravity (CG).

3.1.1 Momenta and Angular Momenta, Newton’s Law

Newton’s Axioms\(^1\) belong to the most important fundamentals in mechanics and had a great impact on the further scientific development at that time. In applied mechanics (where we are far away from the speed of light) these basic laws are still valid today. Especially, the second axiom is one of the most important foundations of kinetics. It says that: *the temporal change of the motion\(^2\) is proportional to the impressed force ...*. This implies that, if no impressed force is present, the motion remains unchanged. The momentum \(p\) is defined by

\[
p = mv
\]

In mathematical terms Newton’s 2. axiom reads as:

\[
F = \frac{dp}{dt} = \frac{d}{dt}(mv)
\]

which is valid in an inertial system. For many applications the mass can be considered constant. For constant mass \(m\) we get:

\[
F = m\frac{dv}{dt} = mg \quad \text{für } m = \text{const.}
\]

\(^1\) Philosophiae Naturalis Prinzipia Mathematica, 1687

\(^2\) Today we would say “momentum”
where $a$ is the absolute acceleration. In integrated form Newton’s law is

$$p - p_0 = \int_{t_0}^{t} F(t^*) \, dt^*$$

(3.4)

The integral of the force over time is called *impulse*. So, we see that change of *momentum = impulse*. Especially, for constant $m$ we get

$$m \left( \vec{v}(t) - \vec{v}(t_0) \right) = p - p_0 = \int_{t_0}^{t} F(t^*) \, dt^*$$

(3.5)

For a particle we can define the *angular momentum* which can be derived by the cross product of the position vector and the momentum:

$$L_0 = \vec{r} \times \vec{p}$$

(3.6)

The angular momentum is related to the origin of the reference frame (as we will see later, any other reference point can be chosen). If we multiply eq. (3.7)

$$\dot{p} = \frac{d}{dt} (mv) = F$$

(3.7)

by the position vector from the left hand side we get:

$$\vec{r} \times \dot{p} = \vec{r} \times \frac{d}{dt} (mv) = \vec{r} \times F$$

(3.8)

On the right hand side of this equation, we see the moment of the impressed force with respect to the origin of the reference frame. The left hand side leads to

$$\frac{d}{dt} \left( \vec{r} \times mv \right) = \vec{r} \times \frac{d}{dt} (mv) + \frac{d}{dt} \left( \vec{r} \times mv \right)$$

(3.9)
and

\[ \frac{d}{dt} \mathbf{r} \times m \mathbf{u} = \mathbf{v} \times m \mathbf{v} = 0 \]

we obtain the relation for the angular momentum and the moment of the impressed force called *angular impulse-momentum principle*:

\[ \frac{d}{dt} \mathbf{L}_0 = \mathbf{M}_0 \quad \text{with} \quad \mathbf{M}_0 = \mathbf{r} \times \mathbf{F} \quad (3.9) \]

Note: While we could derive this relation for a particle from Newton’s law the relation between angular momentum and moment is an independent axiom for rigid bodies. The integrated form is:

\[ \mathbf{L}_0 (t) - \mathbf{L}_0 (t_0) = \int_{t_0}^{t} \mathbf{M}_0 (t^*) \, dt^* \quad (3.10) \]

Instead of cross product in vector notation, in *matrix notation* we can write the angular momentum in the following form:

\[ \mathbf{L}_0 = \tilde{\mathbf{r}} \mathbf{p} \quad (3.11) \]

where the *skew-symmetric* matrix is built up from the components of the position vector

\[ \tilde{\mathbf{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \quad (3.12) \]

Note: Compare this to angular velocities in vector and matrix notation in chapter 2.

### 3.1.2 Rotation of a Body About a Fixed Axis

Although we mainly deal with particles here, the rotational motion of a spinning rigid body of mass \( m \) about a fixed axis can be easily derived from *Newton’s law*. We can find rotational motion about a fixed axis in many machines such as turbines, gears etc.

If we consider a small mass element \( dm \) of the rigid body (fig. 3.2), we can see that it perform a circular motion about the rotation axis (with constant radius \( r \)). From the previous kinematic considerations we know the circular motion causes acceleration (also for constant angular speed). In cylindrical coordinates we got (please compare to chapter B.2):
\[
a = \begin{bmatrix}
a_r \\
a_\phi \\
0
\end{bmatrix} = \begin{bmatrix}
-r\omega^2 \\
r\dot{\omega} \\
0
\end{bmatrix} = \begin{bmatrix}
-r\ddot{\phi} \\
r\dot{\phi}
\end{bmatrix}
\] (3.13)

The (infinitesimal) moment we need to accelerate a single element \(dm\) in circumferential direction (\(\phi\)-direction) is

\[
dM_0 = rdF_\phi
\] (3.14)

According to Newton’s law eq. (3.3) for an infinitesimal small element \(dm\) we can write

\[
dF_\phi = dma_\phi
\] (3.15)

\[
dM_0 = r^2 \ddot{\phi} dm
\] (3.16)

The integration over the whole rigid body yields:

\[
M_0 = \ddot{\phi} \int_{(m)} r^2 dm
\] (3.17)

Or:

\[
M_0 = J_0 \ddot{\phi}
\] (3.18)

with the mass moment of inertia

\[
J_0 = \int_{(m)} r^2 dm
\] (3.19)

with respect to the rotation axis through the origin of the reference frame.

For an arbitrary axis \(x\) through a point \(A\) we get (see fig. 3.3):

\[
M_A = J_A \ddot{\phi}
\]

Figure 3.2: Rotational motion of a rigid body about a fixed axis
with

$$J_A = \int_{(m)} r^2 dm$$

(3.20)

In many cases the mass moment of inertia for a fixed axis $x'-x'$ through center of gravity $S$ is known (e.g. from tables). The transformation for a parallel axis $x-x$ through $A$ can be done by means of Steiner’s theorem

$$J_A = J_S + r_S^2 m$$

(3.21)

As we can see, the mass moment of inertia with respect to an axis through the CG is always the smallest possible one, because the additional Steiner term is always positive.

### 3.1.3 Kinetics of a Particle for Relative Motion

As stated earlier, Newton’s law is valid only when applied in an inertial system. For constant mass and indicating the inertial system by the subscript $I$ we can write:

$$I F = m_I a$$

Now we can replace the absolute acceleration in the $I$-system by the components of the moving $R$-system by using the rotation matrix:

$$r F = A_{RI} I F = m A_{RI} I a = m R a$$

(3.22)

In a next step we express the acceleration by means of the guidance, the Coriolis and the relative acceleration

$$a = a_g + a_{Cor} + a_{rel}$$

Figure 3.3: On the rotational motion about different fixed axes: explanation of geometric quantities
where the subscript $R$ has been omitted for simplicity. Now we introduce this into eq. (3.22):

$$F = ma = m\left(\ddot{a}_g + \ddot{a}_{\text{Cor}} + \ddot{a}_{\text{rel}}\right) \quad (3.23)$$

or after re-arranging the last equation:

$$F_{\text{rel}} = ma_{\text{rel}} = ma - ma_g - ma_{\text{Cor}} \quad (3.24)$$

or

$$F_{\text{rel}} = ma_{\text{rel}} = F - ma_g - ma_{\text{Cor}} \quad (3.25)$$

What can be seen clearly is that we are not allowed to write Newton’s law in its simple form in the moving $R$-system. However, we have to “correct” the formula by the inertia forces: the

Coriolis force $F_{\text{Cor}} = -ma_{\text{Cor}}$

Guidance force and the

guidance force $F_g = -ma_g$

(which consists of three terms and among them the centrifugal force as the most prominent representative of the inertia forces). If we introduce these forces into the last equation, we see that:

$$F_{\text{rel}} = ma_{\text{rel}} = F + F_g + F_{\text{Cor}} \quad (3.26)$$

The appearance of the inertia forces is directly coupled with the shift from the inertial system (where we have no inertia forces) to the moving reference frame $R$. The inertia forces play an important role in D’Alembert’s principle.

Finally we consider the special case that the reference frame is rigidly connected with our mass particle $m$. Then per definition of the $R$-frame we cannot have any relative motion. The relative acceleration and the relative velocity both are zero. The latter implies that the Coriolis force is zero. So we end up formally in a dynamic equilibrium where the sum of all forces including the inertia forces is zero:

$$0 = F + F_g \quad (3.27)$$

3.1.4 Work and Work-Energy Principles

3.1.4.1 Translational Motion of a Particle

The force $F$ moves the particle $m$ along the path. If we consider a small displacement $dr$ we get an infinitesimal contribution of work

$$dW = Fdr = Fd\rho \cos \alpha = F_S ds \quad (3.28)$$
The work is calculated by means of the scalar product of force and displacement, $\alpha$ is the angle between the vectors $\vec{F}$ and $d\vec{r}$. The magnitude of the infinitesimal displacement is $d\vec{r} = ds$. $F_s$ is the component of the force in direction of the tangent to the path. Obviously, the component which is orthogonal to the path tangent cannot contribute to the work of the force.

Integration along the path from $s_0$ to $s_1$ yields the work

$$W = \int_{s_0}^{s_1} F(s) \cos(\alpha(s)) \, ds$$  \hspace{1cm} (3.29)

Next we set $d\vec{r} = vdt$ in eq. (3.28) and $\vec{F}$ is replaced by $ma$ (Newton’s law)

$$F = ma = m \frac{dv}{dt} \quad (m=\text{const.})$$

so that

$$dW = Fdr = mgdv$$  \hspace{1cm} (3.30)

The integration now yields the very important relation

$$W = m \int_{v_0}^{v_1} gdv = \frac{m}{2} (v_1^2 - v_0^2)$$  \hspace{1cm} (3.31)

where the right hand side can be identified as change of the kinetic energy of the particle (mass $m$). The kinetic energy\(^3\) is

$$E_{\text{kin}} = \frac{1}{2} mv^2$$  \hspace{1cm} (3.32)

From eq. (3.31) follows the work-energy theorem:

$$W = E_{\text{kin},1} - E_{\text{kin},0}$$  \hspace{1cm} (3.33)

\(^3\) In the anglo-american literature, frequently the symbol $T$ is used for the kinetic energy.
The work of the external force causes a change of the kinetic energy (increase or decrease depending on the angle between the force and the path).

The left hand side is a path integral which considers what happens along the path, while the energies only take the initial and final state into account.

### 3.1.4.2 Conservative and Non-Conservative Forces, Potential, Energy Theorem

We can distinguish between conservative and non-conservative forces. For conservative forces the value of the path integral eq. (3.29) for a fixed starting and final point is independent of the special shape of the path between these points. This is a very important property of conservative forces.

For conservative forces a scalar function $\Pi$, the potential, exist which allows the calculation of the force vector from the negative gradient of the potential:

$$ F = - \text{grad} \, \Pi $$  

(3.34)

where the negative sign is by definition. For example, in cartesian coordinates we get

$$ F_x = - \frac{\partial \Pi}{\partial x} , \quad F_y = - \frac{\partial \Pi}{\partial y} , \quad F_z = - \frac{\partial \Pi}{\partial z} $$  

(3.35)

**Example:** The potential of the gravitational field of the earth (close to the surface) is

$$ \Pi (x, y, z) = mgz , $$  

(3.36)

where the $z$-axis is orthogonal to the surface and points away from the center of the earth. The calculation of the gradient immediately yields: $F_x = 0, F_y = 0$ und $F_z = -mg$.

The total differential of $\Pi$ is

$$ d\Pi = \frac{\partial \Pi}{\partial x} \, dx + \frac{\partial \Pi}{\partial y} \, dy + \frac{\partial \Pi}{\partial z} \, dz $$  

(3.37)

and on the other hand from eq. (3.28) and eq. (3.29) (in cartesian coordinates) we get

$$ W = \int_0^1 (F_x \, dx + F_y \, dy + F_z \, dz) $$  

(3.38)
If we replace the force components by the derivatives of the gradient and using eq. (3.37), we obtain

\[ W = -\int_0^1 \left( \frac{\partial \Pi}{\partial x} \, dx + \frac{\partial \Pi}{\partial y} \, dy + \frac{\partial \Pi}{\partial z} \, dz \right) = -\int_0^1 d\Pi = -(\Pi_1 - \Pi_0), \quad (3.39) \]

which shows us that the work \( W \) does not depend on the path but only on the values of the potential \( \Pi_0 \) and \( \Pi_1 \) at the initial and final point of the path which proves the independence of the path. The condition that a force field is conservative is the existence of a potential!

In the special case that initial and final point coincide, it follows from eq. (3.39), that \( W = 0 \): if we walk along a closed loop in a conservative force the work resulting from this process is zero!

For non-conservative forces such a potential does not exist. The work integral is path-dependent and the integration along a closed loop yields (in general) a value \( W \neq 0 \).

- gravitational forces,
- elastic forces (e.g. of elastic springs, bending of beams, etc.),
- magnetic forces

Non-conservative forces are

- friction forces or
- forces and moments from external sources.

Finally, for forces having a potential, from eq. (3.33) and eq. (3.39) we can derive the important energy theorem:

\[ W = E_{\text{kin},1} - E_{\text{kin},0} = -(\Pi_1 - \Pi_0) \quad (3.40) \]

from which follows:

\[ E_{\text{kin},1} + \Pi_1 = E_{\text{kin},0} + \Pi_0 \quad (3.41) \]

With the potential energy \( E_{\text{pot}} \) which is corresponding to the value of the \( \Pi \) we get

\[ E_{\text{kin},1} + E_{\text{pot},1} = E_{\text{kin},0} + E_{\text{pot},0} \quad (3.42) \]

This describes the invariance of the total energy (which is the sum of kinetic and potential energy) for a process in conservative fields.
3.1.4.3 Rotation About a Fixed Axis

In analogy to the translational motion of a particle we obtain for the rotation of a rigid body about a fixed axis:

\[
\begin{align*}
\frac{dW}{dt} &= M \varphi \\
W &= \int_{\varphi_0}^{\varphi_1} M(\varphi) \, d\varphi 
\end{align*}
\]  

\[(3.43)\] \quad \[(3.44)\]

Again, the work changes the kinetic energy. For the rotation about a fixed axis we get:

\[
E_{\text{kin}} = \frac{1}{2} J \omega^2
\]

\[(3.45)\]

where \(\omega\) is the angular velocity and \(J\) is the moment of inertia with respect to the spinning axis. We obtain (according to eq. (3.31) and (3.33)):

\[
W = E_{\text{kin},1} - E_{\text{kin},0} = \frac{1}{2} J \left( \omega_1^2 - \omega_0^2 \right)
\]

\[(3.46)\]

A more general analysis of the kinetic energy for the rotation about a free axis will be given later.

3.1.5 Power

Relating the work to time, we come to the concept of power which is very important in all technical disciplines.

*Power unit is 1 W (Watt).*

*Metric horsepower:*

1 hp = 735.5 W

The power \(P\) is mathematically defined by the gradient:

\[
P = \frac{dW}{dt}
\]

\[(3.47)\]

For a translational motion we get

\[
P = F \frac{dr}{dt} = F v \cos(\alpha) = F_s v
\]

\[(3.48)\]

where \(F_s\) is again the tangential component of the force \(F\). For the rotational motion with a fixed axis, the power is

\[
P = M \frac{d\varphi}{dt} = M \omega
\]

\[(3.49)\]
3.2 kinetics of rigid bodies

3.2.1 Momentum of a Rigid Body and the Momentum Theorem

The total mass of a body results from integration over the whole body

\[ m = \int_{(m)} dm = \int_{(V)} \rho dV \quad (3.50) \]

where \( \rho \) is the mass density and \( dV \) is a small volume element. In fig. 3.5, \( P \) is
an arbitrary point of the body, while \( S \) is the center of gravity \(^4\). In the same
way, the resultant of the external forces \( dF \) acting on a small mass element \( dm \) can be obtained by integration over the rigid body \( K \):

\[ \mathbf{F} = \int_{(K)} d\mathbf{F} \quad (3.51) \]

If we consider the contribution of the momentum of a single mass element \( dm \)
(at point \( P \)):

\[ d\mathbf{p} = \mathbf{v} dm \quad (3.52) \]

As the rigid body performs translational and rotational motion as well, we
apply Euler’s formula eq. (2.19) and use the center of gravity \( S \) as reference
point

\[ \mathbf{v} = \mathbf{v}_P = \mathbf{v}_S + \mathbf{\omega} \times \mathbf{r}_{SP} \quad (3.53) \]

If we put this into eq. (3.52) we get

\[ d\mathbf{p} = (\mathbf{v}_S + \mathbf{\omega} \times \mathbf{r}_{SP}) dm = \mathbf{v}_S dm + (\mathbf{\omega} \times \mathbf{r}_{SP}) dm \quad (3.54) \]

Again the total momentum of the rigid body is obtained by integration:

\[ \mathbf{p} = \int_{(m)} \mathbf{v}_S dm + \int_{(m)} (\mathbf{\omega} \times \mathbf{r}_{SP}) dm = \mathbf{v}_S m + \mathbf{\omega} \times \int_{(m)} \mathbf{r}_{SP} dm \quad (3.55) \]

\(^4\) German: “Schwerpunkt” or “Massenschwerpunkt”
The second integral vanished because we have chosen the reference point as the mass center of gravity (good choice!) \(^5\) The momentum then is
\[
p = m \dot{v}_S.
\] (3.56)

It is the product of the mass \(m\) which is concentrated in the center of gravity multiplied by the velocity \(v_S\) of point S. From Newton’s law eq. (3.3) formulated for a small element \(dm\)
\[
t \, dm = dF
\] (3.57)

we obtain after integration
\[
\int (m) \, \ddot{r}dm = \int (m) \, dF = F.
\] (3.58)

In chapter 2 we derived an expression for the acceleration of a point \(P\)
\[
\ddot{r} = \ddot{r}_P = \dot{a}_P = a_S + \dot{\omega} \times r_{SP} + \omega \times (\omega \times r_{SP})
\] (3.59)

putting eq. (3.59) into eq. (3.58) we get (the integrals \(\int (m) \, \mathcal{L}_{RP} \, dm\) are zero):
\[
\int (m) \, \ddot{r}dm = \int (m) \, a_S dm = a_S m
\] (3.60)

The yields the momentum theorem:
\[
\boxed{F = \dot{a}_S m = m \frac{d}{dt} v_S = \dot{p}}
\] (3.61)

where \(m = \text{const.}\) was assumed. We see that the center of gravity (CG) moves as if the force resultant \(F\) would act directly on the CG where the total mass \(m\) is concentrated. The resulting force \(F\) determines the acceleration \(a_s\) of the CG. In general there is an additional rotation of the rigid body, because \(F\) does not act directly on the CG which causes a moment about \(S\).

While we could derive the angular momentum - moment relation from the momentum principle for a mass particle this is not possible for a rigid body. In the latter case we need additional assumptions about the internal forces. Instead the angular momentum theorem was formulated as an independent law for the rigid body by Euler.

3.2.2 Angular Momentum and Moments of Inertia

A small element \(dm\) delivers a contribution

\[\text{compare to. fig. 3.5}\]

\(^5\) Recall that the formula \(\mathcal{L}_{RS} = \frac{1}{m} \int (m) \, \mathcal{L}_{RP} \, dm\) yields the position of the center of gravity \(S\) as vector pointing from the reference point \(R\) to \(S\). If the reference point \(R\) is already identical with \(S\) it follows immediately that \(\mathcal{L}_{RS} = \mathcal{L}_{SS} = 0\).
to the angular momentum (compare to our considerations with respect to a particle) which after integration leads to

\[ L_0 = \int \mathbf{r} \times \mathbf{v} \, dm = \int (\mathbf{r}_S + \mathbf{r}_{SP}) \times \left( \mathbf{v}_S + \mathbf{\omega} \times \mathbf{r}_{SP} \right) \, dm \]  

(3.63)

and multiplying all the terms in brackets of the cross product:

\[ L_0 = \int \mathbf{r}_S \times \mathbf{v}_S \, dm + \int \mathbf{r}_{SP} \times \mathbf{v}_S \, dm + \int \mathbf{r}_S \times (\mathbf{\omega} \times \mathbf{r}_{SP}) \, dm + \int \mathbf{r}_{SP} \times (\mathbf{\omega} \times \mathbf{r}_{SP}) \, dm \]  

(3.64)

The vectors \( \mathbf{r}_S \), \( \mathbf{v}_S \), and \( \mathbf{\omega} \) are constant with respect to the integration over the body and can be moved outside the integral. Now, the second and third term vanish because the integral

\[ \int \mathbf{r}_{SP} \, dm = 0 \]  

(3.65)

as we showed before so that

\[ L_0 = \mathbf{r}_S \times \mathbf{v}_S \, dm + \int \mathbf{r}_{SP} \times (\mathbf{\omega} \times \mathbf{r}_{SP}) \, dm \]  

(3.66)

remains

\[ L_0 = \mathbf{r}_S \times \mathbf{p} + \int \mathbf{r}_{SP} \times (\mathbf{\omega} \times \mathbf{r}_{SP}) \, dm \]  

(3.67)

The first term is the translational part, the second part is the rotational part of the angular momentum:

\[ L_{0,\text{trans}} = \mathbf{r}_S \times \mathbf{p} \]  

(3.68)

\[ L_{0,\text{rot}} = \int \mathbf{r}_{SP} \times (\mathbf{\omega} \times \mathbf{r}_{SP}) \, dm \]  

(3.69)

\[ L_0 = L_{0,\text{trans}} + L_{0,\text{rot}} \]  

(3.70)

We see that also if the body does not rotate we have a contribution to the angular momentum from the translation.

The rotational part can be further investigated and expressed as :

\[ L_{0,\text{rot}} = \int_{(m)} \mathbf{r}_{SP} \times (\mathbf{\omega} \times \mathbf{r}_{SP}) \, dm = -\int_{(m)} \mathbf{r}_{SP} \times (\mathbf{r}_{SP} \times \mathbf{\omega}) \, dm \]  

(3.71)

Splitting the integral terms and the angular velocities, the integral yields the symmetric tensor of the moments of inertia:

\[ J_S = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{bmatrix} \]  

(3.72)
Figure 3.6: Rigid body with reference frame in the center of gravity

The subscript 's' indicates that the inertia tensor is related to center of gravity. The matrix is called moment of inertia matrix or simply inertia matrix. The diagonal elements are called the moments of inertia (with respect to the x-, y- or z-axis) and the off-diagonal terms are the products of inertia or cross products of inertia:

- Moments of Inertia
  \[ J_{xx} = \int (y^2 + z^2) \, dm \]  
  \[ J_{yy} = \int (x^2 + z^2) \, dm \]  
  \[ J_{zz} = \int (x^2 + y^2) \, dm \]

- Products of inertia:
  \[ J_{xy} = J_{yx} = -\int xy \, dm \]
  \[ J_{xz} = J_{zx} = -\int xz \, dm \]
  \[ J_{yz} = J_{zy} = -\int yz \, dm \]

where the coordinates \( x = x_{SP} \), \( y = y_{SP} \) and \( z = z_{SP} \). For a change of the reference frame to another reference frame which has axes parallel to the first one Steiner’s theorem is valid. While the coordinates always appear in quadratic terms in the moments of inertia, the sign of the coordinates are not important, however, in the products of inertia the sign of the coordinates have to be considered.

Now we could write the rotational part of the angular momentum as

\[ \vec{L}_{0, \text{rot}} = \vec{J}_{\text{S} \omega} \]  

(3.74)
then, the total angular momentum is given as
\[ \mathbf{L}_0 = \mathbf{r}_S \times \mathbf{p} + J_S \mathbf{\omega} \] (3.75)

And in matrix notation, the angular momentum is given as
\[ \mathbf{I}\mathbf{L}_0 = \mathbf{I}\mathbf{\hat{r}}_S \mathbf{I}\mathbf{p} + \mathbf{I}\mathbf{J}_S \mathbf{I}\mathbf{\omega} \] (3.76)

3.2.2.1 Principle Axes

If the x-, y- and z-axes are principle axes, the products of inertia (off-diagonal terms) vanish
\[ \mathbf{J}_S = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \] (3.77)

and \( J_1 \) and \( J_3 \) are the maximum and minimum values of the moments of inertia, respectively. The three diagonal moments of inertia are the eigenvalues of matrix in eq. (3.72) and are called principal inertiias. In a basis with symmetric planes of a body, the off-diagonal terms are zero.

Due to the fact that the integration is a linear operator, the inertia-matrix calculation is an additive operation. The inertia matrix of a body with a volume \( V_1 + V_2 \) is the sum of the inertia matrix of volume \( V_1 \) and the inertia matrix of \( V_2 \).

3.2.2.2 Steiner’s Theorem or Parallel Axes Theorem

If we know the inertia matrix with respect to the center of gravity \( S \) we can easily calculate also the inertia matrix for any point, say \( A \), which is the origin of a reference frame which has axes (\( x’\)-\( y’\)-\( z’\)-axes) which are parallel to the \( x\)-\( y\)-\( z\)-axes through \( S \). Steiner’s theorem says that
\[ J_{x’x’} = J_{xx} + (y' S + z'^2 S) \] (3.78a)
\[ J_{y’y’} = J_{yy} + (x'^2 S + z'^2 S) \] (3.78b)
\[ J_{z’z’} = J_{zz} + (x'^2 S + y'^2 S) \] (3.78c)

and
\[ J_{x’y’} = J_{xy} - x' S y' S \] (3.78d)
\[ J_{x’z’} = J_{xz} - x' S z' S \] (3.78e)
\[ J_{y’z’} = J_{yz} - y' S z' S \] (3.78f)
where the coordinates \( x' \), \( y' \), and \( z' \) define the position of \( S \) in the \( x'-y'-z' \) coordinate system in \( A \).

Note: For completeness we note that we can also find the definition of the products of inertia without the negative sign (eq. (3.73d) - eq. (3.73f)) in the literature. But in this case the negative sign is included directly in the \( J_S \), eq. (3.73), where all \( J_{kl}, k, l = x, y, z, k \neq l \) having negative signs.

The rotational part of the angular momentum is

\[
L_{0, \text{rot}} = J_S \omega
\]  
(3.79)

The components of this vector in Cartesian coordinates are

\[
L_{0, \text{rot}, x} = J_{xx} \omega_x + J_{xy} \omega_y + J_{xz} \omega_z \\
L_{0, \text{rot}, y} = J_{yx} \omega_x + J_{yy} \omega_y + J_{yz} \omega_z \\
L_{0, \text{rot}, z} = J_{zx} \omega_x + J_{zy} \omega_y + J_{zz} \omega_z
\]  
(3.80)

or for the principle axes:

\[
L_{0, \text{rot}, 1} = J_1 \omega_1 \\
L_{0, \text{rot}, 2} = J_2 \omega_2 \\
L_{0, \text{rot}, 3} = J_3 \omega_3
\]  
(3.81)

The total angular momentum finally is

\[
L_0 = r_S \times p + J_S \omega
\]  
(3.82)

In matrix notation the angular momentum is

\[
I_L = J_S i p + J_S i \omega
\]  
(3.83)
3.2.3 Angular Momentum Theorem

The theorem of angular momentum introduced by Euler is - as already mentioned- an independent axiom, that cannot be derived from the momentum theorem without further assumption about the internal forces.

The angular momentum theorem is

\[ \dot{L}_0 = M_0 \]  

(3.84)

The external moment changes the angular momentum. For a rigid body the internal forces do not have any influence on the momentum and the angular momentum.

The result of all external moments follows from the integration of all contributions \( r \mathrm{d}F \) over the whole body

\[ M_0 = \int r \times \mathrm{d}F \]  

(3.85)

It is important to note that the reference point 0 of the coordinate frame is arbitrary, however, it must be a fixed point. Angular momentum \( L_0 \) and moment \( M_0 \) must be related to the same point (0).

Finally, the general form of the angular momentum theorem could be written as

\[ \dot{L}_0 = (r_S \times m v_S) + (J_S \omega) = M_0 \]  

(3.86)

The product rule has to be applied. The term

\[ (r_s \times p) = (r_S \times m v_S) = (v_S \times m v_S) + (v_S \times m a_S) \]  

(3.87)

and because \( v_S \times v_S = 0 \) only the second term remains

\[ \dot{L}_0 = (r_S \times m a_S) + (J_S \omega) = M_0 \]  

(3.88)

3.2.4 Change of the Reference Frame

If we move from the fixed reference “0”-system to a moving one “0*-system as shown in fig. 3.8, additional terms describing the motion of the reference frame have to be added. Starting with the position vector

\[ \ell_p = \ell = \ell_{0*} + \ell_{*p} \]  

(3.89)

and the same applies for the velocity vector

\[ \nu_p = \nu = \nu_{0*} + \nu_{*p} \]  

(3.90)
then, the angular momentum could be derived from the integration of eq. (3.62) as follows

\[
L_0 = \int \mathbf{r} \times \mathbf{v}_0 \, dm = \int \left( \mathbf{r}_0^* + \mathbf{r}_P \right) \times \mathbf{v} \, dm = \int \left( \mathbf{r}_0^* + \mathbf{r}_P \right) \times \left( \mathbf{v}_0 + \mathbf{v}_P \right) \, dm
\]

\[
= \int \mathbf{r}_0^* \times \mathbf{v} \, dm + \int \mathbf{r}_P \times \mathbf{v}_0 \, dm + \int \mathbf{r}_P \times \mathbf{v}_P \, dm
\]

\[
= \mathbf{r}_0^* \times \mathbf{p} + \int \mathbf{r}_P \times \mathbf{v}_0 \, dm + \mathbf{L}_0^*
\]

as (static moment)

\[
\int \mathbf{r}_P \, dm = \mathbf{r}_S^* m
\]

we get

\[
L_0 = \mathbf{r}_0^* \times \mathbf{p} + m \mathbf{r}_S^* \times \mathbf{v}_0 + \mathbf{L}_0^*
\]

finally the angular momentum in the moving frame is given as

\[
\mathbf{L}_0^* = \mathbf{L}_0 - \mathbf{r}_0^* \times \mathbf{p} - m \mathbf{r}_S^* \times \mathbf{v}_0
\]  \hspace{1cm} (3.91)

with the momentum of the rigid body given as: \( \mathbf{p} = m \mathbf{v}_S \).

If the moving reference frame origin is the same as the center of gravity of the moving rigid body \( \mathbf{r}_S^* = \mathbf{S} \) such as \( \mathbf{r}_S^* = \mathbf{0} \), the angular momentum could be written as

\[
L_S = \mathbf{L}_0 - \mathbf{r}_0^* \times \mathbf{p}
\]  \hspace{1cm} (3.92)

Using the angular momentum \( \mathbf{L}_0 \) given in eq. (3.82), the angular momentum written at the center of gravity of the moving rigid body is

\[
L_S = \mathbf{J}_S \omega
\]  \hspace{1cm} (3.93)

Figure 3.8: Change of the reference system
As the moment of inertia of the rigid body depends in general on the current position of the body (e.g. angles). This became the main disadvantage of the easy application of the theory. Therefore, the angular momentum is written in terms of body-fixed reference frame, where the dependency of the moments of inertia on time is removed, and the moments of inertia stay constant.

3.2.4.1 Angular Momentum Theorem for Pure Rotation

For a fix-point $A$ only rotation of the body about this point is possible. Euler’s equation for the velocities was

$$\dot{r}_P = \dot{r}_A + \omega \times r_{AP}$$ (3.94)

If $A$ is a fix-point, then $\dot{r}_A = 0$ so that $\dot{r}_P = \omega \times r_{AP}$. In the case that point $P$ is the CG $S$:

$$\dot{r}_S = \dot{r}_S = \omega \times r_S$$ (3.95)

Putting this into eq. (3.86) with $0 = A$:

$$\dot{L}_A = (r_S \times m \omega \times r_S) + (J_S \omega) = M_A$$ (3.96)

The double cross product is

$$r_S \times m \omega \times r_S = \begin{bmatrix} (y_S^2 + z_S^2) m \omega_x - y_S x_S m \omega_y - z_S x_S m \omega_z \\ -y_S x_S m \omega_x + (x_S^2 + z_S^2) m \omega_y - z_S y_S m \omega_z \\ -z_S x_S m \omega_x - z_S y_S m \omega_y + (y_S^2 + x_S^2) m \omega_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$ (3.97)

The matrix can be identified from Steiner’s theorem, see eq. (3.78), so that we can write

$$J_A = J_S + \begin{bmatrix} (y_S^2 + z_S^2) m - y_S x_S m - z_S x_S m \\ -y_S x_S m + (x_S^2 + z_S^2) m - z_S y_S m \\ -z_S x_S m - z_S y_S m + (y_S^2 + x_S^2) m \end{bmatrix}$$ (3.98)

which is the inertia matrix with respect to fix-point $A$. The angular momentum theorem then is

$$\dot{L}_A = (J_A \omega) = M_A$$ (3.100)

As we can see the translational part of the angular momentum is eliminated.
3.2.4.2 Angular Momentum Theorem with Respect to the Center of Gravity

The center of gravity is fixed on the body, but moving in space. Until now, we considered the angular momentum theorem only for an inertial system. Starting our considerations with eq. (3.88):

\[ \dot{L}_0 = (r_S \times ma_S) + (J_S \dot{\omega}) = M_0 \]

we express the moment according to eq. (3.85)

\[ M_0 = \int r \times d\mathbf{F} = \int (r_S + r') \times d\mathbf{F} = r_S \times \mathbf{F} + M_S \]  \hspace{1cm} (3.101)

where \(r'\) is the position vector from point \(S\) to the actual mass element. As can be seen, \(M_0\) can be split into a part of the resultant force (1.part) and a moment about the center of gravity \(M_S\).

\[ \dot{L}_0 = (r_S \times ma_S) + (J_S \dot{\omega}) = r_S \times \mathbf{F} + M_S \]  \hspace{1cm} (3.102)

Now we use the momentum theorem (Newton’s law for a rigid body):

\[ ma_S = \mathbf{F} \]

and we see that eq. (3.102) simplifies to

\[ \dot{L}_S = (J_S \dot{\omega}) = M_S \]  \hspace{1cm} (3.103)

As can be seen the choice of the center of gravity as reference point turns out to be a good choice because only the rotational part of the angular momentum remains.

We should keep in mind that although the reference frames moves with \(S\), the directions of \(x\)-\(y\)-\(z\)-axes are always constant. This means also that if the body

![Figure 3.9: Rigid body (with reference frame in the center of gravity) under external force and moment loading](image-url)
is rotating the moments and products of inertia change with time during the rotational motion. This is a serious disadvantage of this formulation and we have to think about the possibility of a reference frame which is fixed on the body (in $S$).

### 3.2.5 Eulerian Equations, Angular Momentum Theorem in a Rotating Coordinate Frame

As we saw, the rotational motion of a rigid body can be described by the angular momentum theorem. The derivative of the angular momentum with respect to time had to be calculated in a fixed coordinate system, the inertial system.

The extremely big disadvantage of this procedure is that if a body rotates but the coordinate system is fixed you always have to calculate the mass moments of inertia and the products of inertia with respect to this fixed coordinate system but while the body is rotating you may have different moments and products of inertia at every time instant. This makes the practical use of the angular momentum theorem very difficult.

The way out of this problem is to find a moving coordinate system (e.g. a body-fixed coordinate frame) in which the moments and products of inertia remain constant during rotation.

The angular momentum theorem says that:

$$ \dot{L} = M $$

(3.104)

To indicate which coordinate frame is used, we introduce the index $I$:

$$ i\dot{L} = I M $$

(3.105)

The components in the ($I$)-frame can be transformed to the rotating frame ($R$) using the rotation matrix $A_{IR}$ (as we did many times before):

$$ \frac{d}{dt} (A_{IR} R L) = A_{IR} R M $$

(3.106)

or

$$ A_{IR} \dot{R} L + A_{IR} \dot{R} L = A_{IR} R M $$

(3.107)

Premultiplying by $A_{RI}$ from the left side:

$$ A_{RI} \dot{R} L + A_{RI} \dot{A}_{IR} R L = r M $$

(3.108)
In the second expression we recognize the matrix of the angular speeds (compare to chap. 2, kinematics of the rigid body):

\[ R\dot{L} + R\tilde{\omega}_{IR}R\dot{L} = R\dot{M} \]  

(3.109)

The angular speed are the rotational speeds of the rotating frame in components of the \( R \)-frame. The product of the angular velocities and the angular momentum can be written in \textit{classical} formulation as a cross product.

\[ \frac{d'}{dt} (rL) + r\omega \times rL = rM \]  

(3.110)

The first term \( \frac{d'}{dt} (rL) \) is marked by the prime and indicates that the derivative is the temporal change in the rotating frame.

3.2.5.1 \textit{Coordinate System in the Center of Gravity}

As we have seen, the expression for the angular momentum will be much simpler if the reference point is the center of gravity \( S \). In this special case we got:

\[ L_S = J_S \omega \]

Because the moments and products of inertia are constant with respect to a body-fixed rotating coordinate system, the derivative of \( L \) with respect to time yields:

\[ R\dot{L}_S = R_\dot{J}_S R\dot{\omega} \].  

(3.111)

and putting this into eq. (3.109) leads to

\[ R\dot{J}_S R\tilde{\omega} + R\tilde{\omega}_{IR} R\dot{J}_S R\tilde{\omega} = R\dot{M}_S \]  

(3.112)

or shortly, leaving the \( R \)-subscript away:

\[ J_S \dot{\omega} + \tilde{\omega} J_S \omega = M_S \].  

(3.113)

In classical notation this equation writes as

\[ J_S \frac{d'\omega}{dt} + \omega \times J_S \omega = M_S \]  

(3.114)

3.2.5.2 \textit{Body-fixed Principal Axes}

\textit{In principle axes; the moment of inertia matrix has always diagonal form.}

In body-fixed \textit{principal axes}; these equation become very simple in structure.
The inertia matrix then becomes diagonal because the products of inertia are zero:

\[
J_S = \begin{bmatrix}
J_1 & 0 & 0 \\
0 & J_2 & 0 \\
0 & 0 & J_3
\end{bmatrix}
\] (3.115)

It follows that

\[
\begin{align*}
J_1\dot{\omega}_1 - (J_2 - J_3) \omega_2 \omega_3 &= M_1 \\
J_2\dot{\omega}_2 - (J_3 - J_1) \omega_3 \omega_1 &= M_2 \\
J_3\dot{\omega}_3 - (J_1 - J_2) \omega_1 \omega_2 &= M_3
\end{align*}
\] (3.116)

These are the so-called *Eulerian equations* for gyroscopic systems\(^6\). They are three non-linear coupled differential equations. All components of \(J, \omega\) and moment \(M\) have to be written in components of the rotating \((R)\)-coordinate frame. These equations have a great relevance for the derivation of the equations of motion of mechanical systems with rotating components.

### 3.2.6 Angular Momentum Theorem in a Guided Coordinate System

As already mentioned it can be more convenient to use neither a body fixed nor an inertial system to describe the kinematics of a rotating body.

The approach can be clarified with an example. A cylindrical roll is rotating about its rotation axis, which corresponds to the \(x\)-axis (fig. 3.10). The whole system is rotating again about the vertical \(z\)-axis. The guided coordinate system has its origin in the \(z\)-axis. The \(y\)-axis is always horizontal and does not -in contrast to the body fixed \(x'\)-\(y'\)-\(z'\)-system- rotate with the cylindrical roll, whereas \(x\)- and \(x'\)-axis are identical. The coordinate system in the Center of Gravity is -in contrast to the former chapter- not used here.

The angular momentum theorem expressed in the guided coordinate system using the point of origin of the \(R\)-system as centre of reference is

\[
\frac{d'}{dt} (RL_0') + r\vec{\omega}_{GKS} \times RL_0 = RM_0
\] (3.117)

whereas the first term is analogue to eq. (3.109) and eq. (3.110) and describes the relative change of the angular momentum in the guided coordinate system \((GCS)\). The second term of the left hand side is the change of angular momentum due to the rotation of the coordinate system. The expression on the right hand side is the moment expressed in coordinates of the guided \(R\)-system.

---

\(^{6}\) Euler, 1765
In this example the rotation of the guided coordinate system about the vertical rotation axis is expressed by following angular velocity vector

$$R^{\omega}_{GCS} = \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix} \quad (3.118)$$

The mass moments of inertia are:

$$J^R = \begin{bmatrix} J_{xx} & 0 & 0 \\ 0 & J_{yy} & 0 \\ 0 & 0 & J_{zz} \end{bmatrix} \quad (3.119)$$
Due to the symmetry of the cylindrical rolls $J_{zz}$ is equal to $J_{yy}$, the off-diagonal terms are zero. The mass moment of inertia of a homogeneous cylinder (mass $m$, radius $r$, length $l$) is:

$$J_{xx} = \frac{1}{2} mr^2$$

(3.120a)

$$J_{zz} = J_{yy} = \frac{1}{12} m \left( l^2 + 3r^2 \right) + ma^2$$

(3.120b)

The last term in eq. (3.120b) is the parallel axis part. The angular velocity vector of the rotor consists of two components: rotation about $x$-axis and rotation about $z$-axis:

$$R\dot{\omega} = \begin{bmatrix} \omega_x \\ 0 \\ \omega_z \end{bmatrix}$$

(3.121)

The angular momentum is:

$$rL_0 = Jr\dot{\omega} = \begin{bmatrix} J_{xx} \omega_x \\ 0 \\ J_{yy} \omega_z \end{bmatrix}$$

(3.122)

With constant angular velocity in terms of the guided coordinate system

$$\frac{d'}{dt} \left( rL_0 \right) = 0$$

the moment can be determined to be

$$R\dot{M}_0 = \begin{bmatrix} 0 \\ J_{xx} \omega_x \omega_z \\ 0 \end{bmatrix} = -R\dot{M}_{\text{gyroscopic}}.$$  

(3.123)

Figure 3.12: Gyroscopic moment due to the rotation of the cylindrical roll
The **gyroscopic moment** on the right hand side is the action of the rotor to the environment. It loads the bearings as well as other machine parts. In the present example the gyroscopic moment has only a $y$-component. Compared to the guided coordinate system the gyroscopic moment has negative sign, assuming that the two values of the angular velocities are positive.

### 3.3 Kinetic Energy of a Rigid Body

The kinetic energy of a particle is

$$E_{\text{kin}} = \frac{1}{2}mv^2 \quad (3.124)$$

For a rigid body the kinetic energy is obtained by integration over all infinitesimal element of mass $dm$:

$$E_{\text{kin}} = \frac{1}{2} \int_{(m)} v^2 dm \quad (3.125)$$

Using the *Eulerian* kinematic equation (see eq. (2.19))

$$\mathbf{v} = \mathbf{v}_P = \mathbf{v}_A + \mathbf{\bar{\omega}} \cdot \mathbf{r}_{AP},$$

where $A$ is an arbitrary reference point. We express the square of the velocities by:

$$v^2 = \mathbf{v}^T \mathbf{v} = (\mathbf{v}_A + \mathbf{\bar{\omega}} \cdot \mathbf{r}_{AP})^T (\mathbf{v}_A + \mathbf{\bar{\omega}} \cdot \mathbf{r}_{AP}) \quad (3.126)$$

and put it into the integral. Further manipulations lead to

$$E_{\text{kin}} = \frac{1}{2}mv_A^2 + mv_A^T \mathbf{\bar{\omega}} \cdot \mathbf{r}_{AS} + \frac{1}{2} \mathbf{\omega}^T \mathbf{J}_A \mathbf{\omega} \quad (3.127)$$

$\mathbf{r}_{AS}$ is the position vector pointing from point $A$ to the center of gravity $S$. This quantity results from the integral

$$\int_{(m)} \mathbf{r}_{AP} dm = \mathbf{r}_{AS} m$$

and is the static moment which defines the position of the CG.

In vector notation the kinetic energy is

$$E_{\text{kin}} = \frac{1}{2}mv_A^2 + mv_A^T (\mathbf{\omega} \times \mathbf{r}_{AS}) + \frac{1}{2} \mathbf{\omega}^T \mathbf{J}_A \mathbf{\omega} \quad (3.128)$$

We see that kinetic energy has three terms:
• The first is a pure translational part resulting from the velocity of point $A$,

$$E_{\text{trans}} = \frac{1}{2}mv_A^2$$

• The third describes the pure rotation related to point $A$,

$$E_{\text{rot}} = \frac{1}{2}\omega^T J_A \omega$$

• And the middle term is a coupling term of translation and rotation. Don’t forget this part of the kinetic energy.

$$E_{\text{coupling}} = mv_A^T (\omega \times r_{AS})$$

Important Special Cases

1. If the reference point is the center of gravity: $A = S$, then,

$$r_{AS} = r_{SS} = 0$$

the coupling term vanishes, and we get:

$$E_{\text{kin}} = \frac{1}{2}mv_S^2 + \frac{1}{2}\omega^T J_S \omega$$

or simply

$$E_{\text{kin}} = E_{\text{trans}} + E_{\text{rot}}$$

(3.130)

In cartesian coordinates we get

$$E_{\text{trans}} = \frac{1}{2} (v_x^2 + v_y^2 + v_z^2)$$

(3.131)

and

$$E_{\text{rot}} = \frac{1}{2} \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix} \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

(3.132)

$$E_{\text{rot}} = \frac{1}{2} \left( J_{xx} \omega_x^2 + J_{yy} \omega_y^2 + J_{zz} \omega_z^2 + 2J_{xy}\omega_x \omega_y + \ldots \right)$$

$$\ldots + 2J_{xz}\omega_x \omega_z + 2J_{yz}\omega_y \omega_z \right)$$

(3.133)

For principal axes the rotational part of the energy is

$$E_{\text{rot}} = \frac{1}{2} \left( J_1 \omega_1^2 + J_2 \omega_2^2 + J_3 \omega_3^2 \right)$$

(3.134)
2. Rotation about a fix-point $A$: $\mathbf{r}_A = 0$

\[ E_{\text{kin}} = \frac{1}{2} \omega^T J_A \omega \]  

(3.135)

kinetic energy for pure rotation about point $A$.

3. Rotation about a fixed axis through $A$.
The vector of the angular velocity is identical with the spinning axis. The whole expression reduced to one term

\[ E_{\text{kin}} = \frac{1}{2} J_A \omega^2 \]  

(3.136)

where $J_A$ is the moment of inertia with respect to the axis through point $A$.

3.4 Lagrange’s equations of motion of $2^{nd}$ kind

The Lagrangian equations of motion or the Lagrangian equations of the $2^{nd}$. kind (LE2)\(^7\) play a major role in dynamics. They belong to the class of analytical methods which analyze the equations of motions from a global kinetic energy and potential energy consideration. Usually, reaction forces do not appear. They are a standard tool to obtain the equations of motion of dynamic system. The LE2 can be derived from D’Alembert’s principle.

Our system consists of $n$ rigid bodies and has $f$ DOFs. We need the relation between the position vector $\mathbf{r}$ and the $f$ generalized coordinates $q_i$. As shown in chapter eq. (2.8) we can write

\[ \mathbf{r}_i = \mathbf{r}_i (q_1, q_2, \ldots, q_f) \quad i = 1, \ldots, n \]

where $i$ is the subscript of the $i$-th rigid body. The Jacobian-matrix (eq. (2.11)) contains the first derivatives of the position vector with respect to the generalized coordinates

\[ J_i = \left[ \frac{\partial r_{i1}}{\partial q_1} \quad \frac{\partial r_{i2}}{\partial q_2} \quad \cdots \quad \frac{\partial r_{if}}{\partial q_f} \right] \]

The LE2 have the general form:

\[ \frac{d}{dt} \left( \frac{\partial E_{\text{kin}}}{\partial \dot{q}_k} \right) - \frac{\partial E_{\text{kin}}}{\partial q_k} = Q_k \quad k = 1, \ldots, f \]  

(3.137)

\(^7\) There are also Lagrange’s equations of the first kind, which we do not treat here. These equations contain the constraints explicitely, while the equations of the second kind work with the generalized coordinates which already contain the constraints implicitely.
where \( E_{\text{kin}} \) is the total kinetic energy of the whole system which is the sum of the kinetic energies of all \( n \) rigid bodies:

\[
E_{\text{kin}} = \sum_{i=1}^{n} E_{\text{kin},i}
\]  

The eq. (3.137) yields \( f \) differential equations according to the number of the degrees of freedom \( f \), respectively the number of generalized coordinates. The \( Q_k \) are generalized forces. Which result from the external impressed forces from all \( n \) rigid bodies.

The generalized force \( Q_k \) follows from the forces and moments (\( l \)-th force/moment at the \( i \)-th body)

\[
Q_k = \sum_{i=1}^{n} \left( \sum_{l=1}^{L_i} \left( \frac{\partial r_{il}}{\partial q_k} \right)^T E_{il}^{(c)} + \sum_{l=1}^{L_i} \left( \frac{\partial \phi_{il}}{\partial q_k} \right)^T M_{il}^{(c)} \right)
\]

by a projection of a force or moment with the help of the Jacobian matrix. This is done in such a way that only the part of the force/moment is considered that contributes to the motion of the system (to the \( k \)-th gen. coordinate) according to the constraints\(^8\).

### 3.4.1 Conservative Systems

Generally, for conservative forces a potential exists (see chapter 3.1.4.2), which is not the case for non-conservative forces. Thus, if the generalized force results from conservative forces/moments we can derive them directly from the potential energy

\[
Q_k = Q_{k,\text{kons}} = -\frac{\partial E_{\text{pot}}}{\partial q_k}
\]

As shown earlier, conservative forces can be calculated from the potential by differentiation of \( E_{\text{pot}} = E_{\text{pot}}(q_1, \ldots, q_f) \) but now with respect to the \( k \)-th generalized coordinate. Introducing eq. (3.140) into eq. (3.137) we get

\[
\frac{d}{dt} \left( \frac{\partial E_{\text{kin}}}{\partial \dot{q}_k} \right) - \frac{\partial E_{\text{kin}}}{\partial q_k} = -\frac{\partial E_{\text{pot}}}{\partial q_k}
\]

with the total energies

\[
E_{\text{kin}} = \sum_{i=1}^{n} E_{\text{kin},i}
\]

\(^8\) example is given in the class room
and

\[ E_{\text{pot}} = \sum_{i=1}^{n} E_{\text{pot},i}. \]

Introducing the *Lagrangian* function \( L^9 \) (also called kinetic potential)

\[ \mathcal{L} = E_{\text{kin}} - E_{\text{pot}} \]  \hfill (3.142)

we see that

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad k = 1, \ldots, f. \]  \hfill (3.143)

which are the famous Langrangian equations of the 2. kind.

### 3.4.2 Conservative and Non-conservative Forces, Rayleigh Energy Dissipation Function

In this case a potential exists only for the conservative forces/moments and only these forces/moments can be treated by potential energy terms. The non-conservative forces/moments have to be considered by the generalized forces \( Q_k = Q_{k,\text{nc}} \):

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} = Q_{k,\text{nc}} \quad k = 1, \ldots, f \]  \hfill (3.144)

However, some types of non-conservative forces/moments (like viscous damping forces) can be expressed by an energy dissipation function \( D \) which has the dimension of a power. This dissipation function is called *Rayleigh dissipation function* and can be defined by a general expression

\[ D = \frac{1}{2} \sum_i \sum_j c_{ij} \dot{q}_i \dot{q}_j \]  \hfill (3.145)

where the the \( \dot{q} \)'s are the time derivatives of the generalized coordinates \( q \) and the \( c \)'s are coefficients. The non-conservative forces can be derived from \( D \) in a similar way as the conservative forces, namely by differentiating a scalar function:

\[ Q_{k,\text{nc}} = -\frac{\partial D}{\partial \dot{q}_k} \]  \hfill (3.146)

A confusion of the *Lagrangian* function \( \mathcal{L} \) with the angular momentum \( L \) is not expected here.
but here we differentiate with respect to a velocity. This leads to an extension of the Lagrangian equations

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_{k,nc} - \frac{\partial D}{\partial \dot{q}_k} \quad k = 1, \ldots, f \tag{3.147}
\]

Here the \(Q\)'s only contain those forces which cannot be expressed by \(D\). As an example we can consider a viscous damper with damping constant \(c\). The motion of the left attachment point of the damper is described by \(q_l\) and the right attachment point by \(q_r\), both having the same direction. The dissipation function for this case is

\[
D = \frac{1}{2} c v_{rel}^2 = \frac{1}{2} c (\dot{q}_r - \dot{q}_l)^2
\]

As we can see this example also leads to the general form of eq. (3.145):

\[
D = \frac{1}{2} c (\dot{q}_r - \dot{q}_l)^2 = \frac{1}{2} c (\dot{q}_r \dot{q}_r - 2 \dot{q}_l \dot{q}_r + \dot{q}_l \dot{q}_l)
\]

The damping force is obtain by differentiation

\[
F_{\text{damper},r} = - \frac{\partial D}{\partial \dot{q}_r} = - c (\dot{q}_r - \dot{q}_l)
\]
\[
F_{\text{damper},l} = - \frac{\partial D}{\partial \dot{q}_l} = c (\dot{q}_r - \dot{q}_l)
\]

### 3.5 Equations of Motion of a Mechanical System

From the Lagrangian equations or Newton/Euler's equations we get the differential equations describing the motion of a mechanical system with \(f\) DOF. They have the general form

\[
M(q) \ddot{q} + h(q, \dot{q}, t) = Q_{nc}(t) \tag{3.148}
\]
where:

- $M(q) f \times f$ mass matrix (which in general is depending on the generalized coordinates $q$)
- $Q_{nc}(t)$ vector of time dependent external forces and moments (dimension $f$)
- $h(q, \dot{q})$ vector of forces/moments (dimension $f$) depending on the generalized coordinates and/or the generalized velocities and/or time $t$, e.g.
  - Conservative elastic forces, depending on $q$.
  - Dissipative forces, e.g. from viscous damping, friction, depending on $\dot{q}$.
  - Gyroscopic moments, depending on velocities $\dot{q}$.
  - ...

### 3.5.1 Linearization of the Equations of Motion

In many applications, the motion described by $q(t)$ can be splitted into a reference motion $q_r(t)$ which is the desired motion of the system and a more or less small disturbance $\Delta q(t)$ (e.g. a vibration about this reference motion $q_r(t)$).

$$q(t) = q_r(t) + \Delta q(t) \quad (3.149)$$

The same can be done with the external forces which are split into reference forces and disturbances

$$Q(t) = Q_r(t) + \Delta Q(t) \quad (3.150)$$

with

$$F(\ddot{q}_r, \dot{q}_r, q_r, t) = M(q)\ddot{q} + h(\dot{q}, q, t) = Q(t) \quad (3.151)$$

we can derive a linearized differential equation for the motion around the desired trajectory $q_r(t)$ by using the Taylor series which we truncate after the linear part:

$$F(\ddot{q}_r + \Delta \ddot{q}, \dot{q}_r + \Delta \dot{q}, q_r + \Delta q, t) = F(\ddot{q}_r, \dot{q}_r, q_r, t) + \frac{\partial F}{\partial \ddot{q}} \bigg|_r \Delta \ddot{q} + \ldots$$

$$\ldots + \frac{\partial F}{\partial \dot{q}} \bigg|_r \Delta \dot{q} + \frac{\partial F}{\partial q} \bigg|_r \Delta q \quad (3.152)$$

Splitting the last equation into reference motion (which can also be a static displacement) and disturbance (which is a small motion around the reference trajectory), we get
• **Reference motion**

\[
M(q_r)\ddot{q}_r + h(\dot{q}_r, q_r, t) = Q_r(t) \tag{3.153}
\]

• **Disturbance**

\[
[M(q_r)\Delta \ddot{q} + \left. \frac{\partial h}{\partial q} \right|_r \Delta \dot{q} + \left. \frac{\partial h}{\partial \dot{q}} \right|_r \Delta q + \left. \frac{\partial (M(q)\dot{q})}{\partial q} \right|_r \Delta q] = \Delta Q(t) \tag{3.154}
\]

### 3.5.2 Equation of Motion of a Linear Time-Variant and Time-Invariant Mechanical System

If the linearization is done with respect to a reference trajectory \(q_r(t)\) the resulting system matrices are time dependent. We call this a *time-variant* system.

\[
M(q_r)\Delta \ddot{q} + \{C(t) + G(t)\} \Delta \dot{q} + \{K + N\} \Delta q = \Delta Q(t) \tag{3.155}
\]

If the linearization is done with respect to a reference point \(q_r\) (e.g. the static equilibrium position due to the gravitational forces) the resulting system matrices are time independent and the system is called *time-invariant*. The system matrices are

\[
M\Delta \ddot{q} + \{C + G\} \Delta \dot{q} + \{K + N\} \Delta q = \Delta Q(t) \tag{3.156}
\]

The \(f \times f\) matrices of the equation of motion are:

- **\(M\)** mass matrix
- **\(C\)** symmetric damping matrix (describing velocity dependent damping forces/moments)
- **\(G\)** skew-symmetric gyroscopic matrix (describing velocity dependent gyroscopic moments)
- **\(K\)** symmetric stiffness matrix (describing position dependent restoring forces/moments)
- **\(N\)** skew-symmetric matrix of non-conservative position dependent forces/moments
3.6 STATE SPACE REPRESENTATION OF A MECHANICAL SYSTEM

3.6.1 The General Non-linear Case

The equation of motion

\[ M(q) \ddot{q} + h(\dot{q}, q, t) = Q(t) \]  

(3.157)

is in general non-linear and contains the second order derivatives with respect to time as highest order (due to the fact the acceleration appears in the equations). However, many numerical algorithms\(^\text{10}\) for ordinary differential equations (ODE) and theories in control make use of first-order formulations.

\[ \dot{z} = f(z, t) \]  

(3.158)

which is a first-order differential equation. The second-order equation of motion eq. (3.157) has to be converted into a first-order equation which is done by the following intermediate step. We introduce the state space vector \( \tilde{z} \):

\[ \tilde{z} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \]  

(3.159)

which contains the generalized coordinates and the velocities as well. The state of a mechanical system is defined by the generalized coordinates (displacements or angles) and the velocities!

The acceleration comes into the play when the derivative of eq. (3.159) is formulated:

\[ \dot{\tilde{z}} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} \]  

(3.160)

Comparing eq. (3.160) and eq. (3.159), we see that

\[ \dot{z}_1 = \dot{z}_2 \]  

(3.161)

If we use \( \tilde{z} \) instead of the \( q \)'s, eq. (3.157) becomes

\[ M(\tilde{z}_1) \ddot{z}_2 + h(\tilde{z}_1, \tilde{z}_2, t) = Q(t) \]  

(3.162)

so that we only have first-order time derivatives. However we have to pay the price that the number of the differential equations doubled and has dimension \( 2f \) now.

\(^{10}\) Examples are the Euler- (very simple but inaccurate algorithm), the Runge-Kutta- or the Adams-method
Converting eq. (3.161) and eq. (3.162) into eq. (3.158) we get:

\[
\dot{\bar{z}} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \frac{\dot{z}_2}{M^{-1}(\bar{z}_1)} \\ Q(t) - h(\bar{z}_1, \bar{z}_2, t) \end{bmatrix} = f(\bar{z}, t) \tag{3.163}
\]

For the solution of the problem we also have to define the initial \textit{positions} and the initial \textit{velocities}:

\[
\bar{z}(t = 0) = z_0 = \begin{bmatrix} q_0 \\ \dot{q}_0 \end{bmatrix} \tag{3.164}
\]

Usually, in a state-space representation, a second equation called the measurement equation, is used. The \textit{measurement equation} links the measured quantities \( y \) with state-space variables in \( \bar{z} \):

\[
y = g(\bar{z}, t) \tag{3.165}
\]

This is a very general (non-linear) formulation. As an example a measured quantity \( y_1 \) could be a strain which has to be related to displacements \( q \).

### 3.6.2 The Linear Time-Invariant Case

In many practical cases it is possible to linearize the non-linear equations of motion or they are linear from the beginning. The linear theory is well examined and many important theorems are available for linear systems.

If the mechanical system behaves linearly, the equation of motion is (the \( \Delta \) is left away here for simplicity)

\[
M\ddot{q} + (C + G)\dot{q} + (K + N)q = Q(t) \tag{3.166}
\]

In this case the state-space equation \( \dot{\bar{z}} = f(\bar{z}, t) \) becomes

\[
\dot{\bar{z}} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \frac{\ddot{z}_2}{M^{-1}(\bar{z}_1)} \\ Q(t) - (C + G)\dot{q} - (K + N)q \end{bmatrix} = f(\bar{z}, t) \tag{3.167}
\]

This can be re-organized in the following way:

\[
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{I}{M} \\ -M^{-1}(K + N) & -M^{-1}(C + G) \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}Q(t) \end{bmatrix} \tag{3.168}
\]

or

\[
\begin{bmatrix} \dot{q} \\ \dot{\bar{y}} \end{bmatrix} = \begin{bmatrix} 0 & \frac{I}{M} \\ -M^{-1}(K + N) & -M^{-1}(C + G) \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{\bar{y}} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}Q(t) \end{bmatrix} \tag{3.169}
\]
The general formulation of a linear state space model in control theory has the form

\[ \dot{z} = Az + Bu \]  \hspace{1cm} (3.170)

where

- \( A \): 2\( f \times 2\( f \) system matrix
- \( B \): 2\( f \times \) input matrix
- \( u \): input (here the generalized forces \( u(t) = Q(t) \))

The two matrices can be identified as:

\[
A = \begin{bmatrix}
    0 & I \\
    -M^{-1}(K+N) & -M^{-1}(C+G)
\end{bmatrix} \hspace{1cm} (3.171)
\]

\[
B = \begin{bmatrix}
    0 \\
    M^{-1}
\end{bmatrix} \hspace{1cm} (3.172)
\]

Assume we have measured \( n_y \) quantities which we arrange in the vector \( y \). The corresponding linear measurement equation is

\[ y = C_{\text{meas}}z + D_{\text{meas}}u(t) \hspace{1cm} (3.173) \]

with

- \( C_{\text{meas}} \): measurement or output matrix (size \( n_y \times 2f \))
- \( D_{\text{meas}} \): transmission matrix (size \( n_y \times f \))

The eigenvalues \( \lambda \) of the system matrix \( A \) are the system poles and they contain important information about the resonant frequencies and damping behaviour. If one of the real parts of the \( \lambda \)'s is positive, the system is unstable and large amplitude can occur so that the system can be destroyed. More about these relations will be given in the chapters dealing with vibrations of mechanical systems.

### 3.6.3 The Linear Time-Invariant Case in Discrete Time

Dealing with digital techniques, we have sampled time series and the values are only available at time instants \( t_k, k = 1, 2, 3, \ldots \)

Assuming that \( u \) is constant between two sample times and transforming eq. (3.170) from discrete time to continuous time yields:

\[ \dot{z}(k+1) = A_{\text{dis}}z(k) + B_{\text{dis}}u(k) \hspace{1cm} (3.174) \]
The measurement equation can be written as

\[ y(k) = C_{meas}z(k) + D_{meas}u(k) \]  

(3.175)

where the discrete-time matrices can be calculated from the continuous-time matrices \( A \) and \( B \):

\[
A_{dis} = e^{A \Delta t} \quad (3.176)
\]

\[
B_{dis} = \int_0^{\Delta t} e^{A(\Delta t - \tau)} B \, d\tau \quad (3.177)
\]

This can be shown by solving the differential equation\(^{11}\). The matrices of the measurement equations remain unchanged, because the measurement equation is an algebraic equation and contains no differential expression.

LINEAR VIBRATIONS OF SYSTEMS WITH ONE DEGREE OF FREEDOM

4.1 GENERAL CLASSIFICATION OF VIBRATIONS

Different possibilities for a classification exist\(^1\):

- Classification with respect to the number of degrees of freedom: \( f \)
  - 1 DOF or known as Single Degree Of Freedom (SDOF)
  - \( n \) DOFs or known as Multiple Degrees Of Freedom (MDOF)
  - infinite number of DOFs (continuous system: beams, plates, ...)

Spatially discrete vibration systems with \( f = n \) DOFs can result from discrete multi-body systems but from finite-element systems, where the continuum was discretized to some representative nodal points leading to a finite number of DOFs.

- Classification with respect to the character of the describing equation of motion:
  - linear vibrations
  - non-linear vibrations (e.g. oscillators with non-linear spring or damper characteristics, contact or play, etc.)

Even for an apparently simple non-linear system with one DOF the analytical solution can be very complicated, if not impossible. In the latter case, a solution is only possible by numerical methods.

As shown in the last chapter non-linear systems can be linearized about a reference trajectory or a reference point in many cases. The description by linear differential equations has many methodological advantages.

In the context of non-linear differential equations we can get new, very complex dynamic behaviour, which does not occur with linear systems. An interesting phenomenon is deterministic chaos. Although the motion of the system is described by deterministic equations and the input excitation is deterministic (e.g. a sine-excitation) the response of the system seems to be random (which we call chaotic in this context). Starting with two adjacent initial conditions which differ only slightly we may end up in totally different system states. This phenomenon is called the "butterfly effect". However, order can be found when we observe the system over a very long time span. The system converges to a so-called 'strange attractor'.

- Classification with respect to the origin of the vibration
  - free vibrations
  - self-excited vibrations
  - parameter-excited vibrations
  - forced vibrations
  - coupled vibrations

If the system performs free vibrations (linear or non-linear) it starts from an initial state and is left alone, there is no other influence from outside the system.

**Self-excited vibrations** can occur if the system has access to an external reservoir of energy other than forced vibrations where the rhythm of the excitation is prescribed, here the system itself determines the rhythm of the energy transfer into the system. The system excites itself. It takes as much energy as it needs to maintain the vibration (e.g. as in the case of a pendulum clock where the energy storage is an elastic spring). These systems are called autonomous systems. If more energy per cycle flows into the system than energy is dissipated due to friction etc. then unstable vibration can occur leading to strongly increasing amplitudes as in the case of unstable feedback control loops. This must be avoided under any conditions.
This may also occur if there is an interaction of a structure with a fluid as in the case of an airplane wing which tends to unstable vibration when reaching a critical speed. This phenomenon is called "wing flutter". Usually, this speed is beyond the travelling speed of the airplane, but has to be checked for every new prototype.

Also the destruction of the Tacoma-Bridge in the 40th in the US is a "good" example how dangerous self-excited vibrations by fluid-structure interaction can be. Other examples are unstable vibrations of turbo machinery coming from or friction induced vibrations in brakes (squeal noise) or slip-stick phenomena with tool machines leading to marks of the cutting tool at the surface of the workpiece. Also unstable motion of the bogie of rail vehicles at very high speeds belong to the group of self-excited vibrations.

**Parameter-excited vibrations** occur if one or more coefficients of the differential equations are not constant but periodically time-varying. The frequency of the parameter change is prescribed explicitely as a function of time, e.g. by the rotational speed of a shaft. Examples are: pendulum with periodically varying length, rotating shaft with unsymmetric cross-section, periodically varying stiffness of gear-wheels.

**Forced vibrations** emerge from external disturbances (e.g. periodical disturbances from unbalance). The rhythm of the vibration here is not prescribed by varying parameters but by a time-dependent disturbance term on the right hand side of the equation of motion.

**Coupled oscillations** occur by the fact that two oscillators can influence and excite each other.

All these origins for vibrations can occur also in combined form.

Another possibility of classification is due to the character of the excitation:

- deterministic
  - harmonic / periodic
  - transient (e.g. vibrations after an impact)
- stochastic (or random) vibrations
  - stationary
— instationary

4.2 FREE UNDAMPED VIBRATIONS OF THE LINEAR OSCILLATOR

4.2.1 Equation of Motion

Newton’s resp. Euler’s equations or the Lagrange’s formalism can be used here in order to derive the equation of motion. The example shows a mass \( m \) with elastic foundation (total stiffness \( k \))

*Newton’s law:*

\[
m\ddot{x} = \sum_i F_i = -\frac{1}{2}k x
\]

\[
m\ddot{x} + kx = 0
\]

Division by the mass yields:

\[
\ddot{x} + \omega_0^2 x = 0
\]

with

\[
\omega_0 = \frac{k}{m}
\]

\( \omega_0 \) is the natural circular frequency of the free undamped vibration.

---

**Figure 4.1: Vibration system with one DOF**
4.2 Free Undamped Vibrations of the Linear Oscillator

4.2.2 Solution of the Equation of Motion

This differential equation has the solution:

\[ x(t) = A_c \cos \omega_0 t + A_s \sin \omega_0 t \]  \hfill (4.5)

The constants \( A_c \) und \( A_s \) follow from the initial conditions:

\[ x_0 = x(t = 0) \] \hfill (4.6)
\[ v_0 = \dot{x}(t = 0) \] \hfill (4.7)

For \( t = 0 \) this leads directly to \( A_c = x_0 \) and after differentiation of eq. (4.5) to get the velocities, the constant \( A_s = \frac{v_0}{\omega_0} \) so that we can express the constants in terms of the initial displacement and the initial velocity:

\[ x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t \] \hfill (4.8)

Further important quantities are the time of a cycle or period:

\[ T_0 = \frac{2\pi}{\omega_0} \] \hfill (4.9)

And the natural frequency:

\[ f_0 = \frac{1}{T_0} = \frac{\omega_0}{2\pi} \] \hfill (4.10)

Another kind of representation of the solution is:

\[ x(t) = A \sin (\omega_0 t + \varphi) \] \hfill (4.11)
Using trigonometric theorems:
\[
\sin (\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha .
\]  
(4.12)

Here:
\[
x(t) = A (\sin \omega_0 t \cos \varphi + \sin \varphi \cos \omega_0 t)
\]
(4.13)
\[
= A \cos \varphi \sin \omega_0 t + A \sin \varphi \cos \omega_0 t
\]
(4.14)
with
\[
A_c = A \sin \varphi
\]
(4.15)
\[
A_s = A \cos \varphi
\]
(4.16)

Furthermore:
\[
A_c^2 + A_s^2 = A^2 \left( \sin^2 \varphi + \cos^2 \varphi \right)
\]
(4.17)
and
\[
\frac{A_c}{A_s} = \frac{A \sin \varphi}{A \cos \varphi}
\]
(4.18)
so that
\[
A = \sqrt{A_c^2 + A_s^2}
\]
(4.19)
and
\[
\tan \varphi = \frac{A_c}{A_s}
\]
(4.20)

Figure 4.3: Phase Angle \(\varphi\)

4.2.3 Complex Notation

Starting with the equation of motion (eq. (4.3))
\[
\ddot{x} + \omega_0^2 x = 0
\]
(4.21)
and solve it using an exponential approach:

\[ x(t) = e^{\lambda t} \]  
(4.22)

\[ \dot{x}(t) = \lambda e^{\lambda t} \]  
(4.23)

\[ \ddot{x}(t) = \lambda^2 e^{\lambda t} \]  
(4.24)

we get

\[ \lambda^2 e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0 \]  
(4.25)

\[ \lambda^2 + \omega_0^2 = 0 \]  
(4.26)

\[ \lambda^2 = -\omega_0^2. \]  
(4.27)

which has the two conjugate complex solutions \( \lambda \):

\[ \lambda_{1,2} = \pm i\omega_0 \]  
(4.28)

The general solution for \( x(t) \) is:

\[ x(t) = A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t} \]  
(4.29)

where the constants \( A_+ \), \( A_- \) are conjugate complex, too, so that the solution \( x(t) \) becomes real again. The constants can be derived from the initial displacement and velocity, respectively.

4.2.4 Relation Between Complex and Real Notation

We start with the well-known general formula:

\[ e^{\pm i\lambda} = \cos \lambda \pm i \sin \lambda \]  
(4.30)

which we can use to express \( \sin \) and \( \cos \) by:

\[ \cos \lambda = \frac{e^{i\lambda} + e^{-i\lambda}}{2} \]  
(4.31)

\[ \sin \lambda = \frac{e^{i\lambda} - e^{-i\lambda}}{2i} \]  
(4.32)

Putting this into eq. (4.29) yields:

\[ x = \frac{A_c}{2} (e^{i\omega_0 t} + e^{-i\omega_0 t}) + \frac{A_s}{2i} (e^{i\omega_0 t} - e^{-i\omega_0 t}) \]  
(4.33)

\[ = \frac{A_c}{2} (e^{i\omega_0 t} + e^{-i\omega_0 t}) - \frac{i A_s}{2} (e^{i\omega_0 t} - e^{-i\omega_0 t}) \]  
(4.34)

\[ = \frac{A_c - i A_s}{2} e^{i\omega_0 t} \]  
(4.35)

\[ + \frac{A_c + i A_s}{2} e^{-i\omega_0 t} \]  
(4.35)
complex conjugate of $A$ is $A^*$

where $(...)^*$ denotes the conjugate complex number of $(...)$. 

This delivers the relation between the real constants $A_c$, $A_s$ and the complex constants $A_+$ and $A_-$:

\[
A_+ = \frac{A_c - iA_s}{2} = A^*_c
\]

\[
A_- = \frac{A_c + iA_s}{2}
\]

Furthermore, it follows that:

\[
\text{Re}\left\{A_+ e^{i\omega_0 t}\right\} = \frac{1}{2} (A_c \cos \omega_0 t + A_s \sin \omega_0 t)
\]

\[
\text{Re}\left\{A_- e^{-i\omega_0 t}\right\} = \frac{1}{2} (A_c \cos \omega_0 t + A_s \sin \omega_0 t)
\]

so that

\[
\text{Re}\left\{A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t}\right\} = (A_c \cos \omega_0 t + A_s \sin \omega_0 t)
\]

\[
\text{Im}\left\{A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t}\right\} = 0
\]

4.2.5 Further Examples of Single Degree of Freedom Systems

There are many applications, where the elastic element of a vibration system is not a simple spring but e.g. an elastic beam. The lowest natural frequency of such a system can be calculated by a relatively simple approximation, where we assume that the deflection shape of the vibrating beam or shaft etc. is approximately equal to the static deflection shape (which is approximately true for the lowest vibration mode).

**Example**  Flexural Vibration of a Beam:

The mass of the beam is assumed to be small: $m >> m_B$
The stiffness constant is \( k = \frac{48EI}{l^3} \) where \( E \) is the Young’s modulus, \( I \) is the area moment of inertia. The constant follows from the elementary theory of beam bending, where the deflection \( x \) of the beam due to a static force \( F \) in the middle of the beam is \( x = \frac{F l^3}{48EI} \). The stiffness coefficient \( k \) follows immediately from \( F = kx \). The natural circular frequency is:

\[
\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{48EI}{ml^3}}
\]  

(4.42)

**Example**  Torsion Vibration of a Disc:

The equation of motion is:

\[
J \ddot{\varphi} + k \dot{\varphi} = 0
\]  

(4.43)

Here the stiffness coefficient of the shaft is \( \tilde{k} = \frac{GIp}{l} \), where \( G \) is the shear modulus and \( Ip \) is the polar moment of inertia. A static torque \( M_T \) at the position of the disc will cause a twist angle \( \varphi \) of the shaft:

\[
\omega_0 = \sqrt{\frac{k}{J}} = \sqrt{\frac{GIp}{Jl}}
\]  

(4.44)

Here, again the mass moment of inertia of the bar has been neglected.

### 4.2.6 Approximate Consideration of the Spring Mass

If the spring mass (which can be the mass of a beam, bar or shaft depending on the actual case) cannot be neglected compared to the main mass \( m \), we can consider the effect of the spring mass on the natural frequency.

![Torsion vibration of a disc](image)  

Figure 4.5: Torsion vibration of a disc
**Example**  Mass $m$ connected to a bar:

The bar has the mass $m_S$, length $l$, a cross section area $A$ and Young’s modulus $E$. The stiffness coefficient is $k = \frac{EA}{l}$.

The real deflection shape is approximated by the static deflection shape. At a certain position $z$ of the bar we have a static deflection:

$$ u(z, x) = \frac{z}{l} x $$

where $x$ is the deflection of the mass $m$. At the top $z = 0$ and hence $u = 0$, at $z = l$ we get $u = x$. Now we consider the kinetic energy of both masses:

$$ E_{kin} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \int_0^l u^2 \, dm = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \int_0^l \left( \frac{z}{l} \dot{x} \right)^2 \rho A d z $$

$$ = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \rho A \frac{1}{l^2} \int_0^l z^2 d z = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \rho A \frac{1}{l^2} \frac{1}{3} l^3 $$

The mass of the bar is $m_S = \rho Al$ with $\rho$ as the mass density so that

$$ E_{kin} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \left( \frac{1}{3} m_S \right) \dot{x}^2 $$

Finally, we get:

$$ E_{kin} = \frac{1}{2} \left( m + \frac{m_S}{3} \right) \dot{x}^2 = \frac{1}{2} (m_{eff}) \dot{x}^2 $$

As we can see (for this example) that the mass of the bar is considered by one third of its total mass.

The natural frequency is:

$$ f_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m + \frac{m_S}{3}}} = \frac{1}{2\pi} \sqrt{\frac{EA/l}{m + \frac{m_S}{3}}} $$
which is smaller as in the case when we neglect the mass of the bar.

The factor $\frac{1}{3}$ applies also in the case of mass connected with a helical spring, or in the case of a disc mounted on a torsion bar. In the case of flexural bending of beams the static deflection shape is more complicated as in this example but can be calculated by means of the beam theory or taken from tables. The general procedure to calculate the kinetic energy is identical to the example.

### 4.3 Free Vibrations of a Viscously Damped Oscillator

#### 4.3.1 Equation of Motion

We consider viscous damping which means that the damper force is proportional to the relative velocity of the two connection pins of the damper. The force represents fairly well the conditions of damping due to the oil in a dashpot. Other, more complicated damping laws can be considered which lead to non-linear equations of motion. Non-linear Equation Of Motion (EOM) (EOM) require a much more complicated mathematical treatment. In most cases only a numerical solution is possible.

The free body diagram and application of Newton’s law yields:

$$m\ddot{x} = \sum_i F_i = -2k\frac{x}{2} - c\dot{x}$$  \hspace{1cm} (4.51)
or

\[ m\ddot{x} + c\dot{x} + kx = 0 \]  \hspace{1cm} (4.52)

The mass \( m \), the viscous damping coefficient \( c \) and the stiffness \( k \) are constants (as in the previous case) and \( m, c, k > 0 \). The viscous damping reduces the total energy of the vibrating system. Division by \( m \) yields:

\[ \ddot{x} + 2\delta \dot{x} + \omega_0^2 x = 0 \]  \hspace{1cm} (4.53)

where

\[ \delta = \frac{c}{2m} \]  \hspace{1cm} (4.54)

and

\[ \omega_0 = \sqrt{\frac{k}{m}} \]

Transition to the dimensionless time (or an angle, depending on the interpretation)

\[ \tau = \omega_0 t \]  \hspace{1cm} (4.55)

leads to the following derivatives (using the chain rule):

\[ \frac{d \left( \ldots \right)}{dt} = \frac{d \left( \ldots \right)}{d\tau} \frac{d\tau}{dt} = \frac{d \left( \ldots \right)}{d\tau} \omega_0. \]  \hspace{1cm} (4.56)

If we distinguish the derivatives with respect to time \( t \) by the dot, and the derivatives with respect to \( \tau \) by the prime, we find that:

\[ \omega_0^2 \dddot{x} + 2\delta \omega_0 \dot{x} + \omega_0^2 x = 0. \]  \hspace{1cm} (4.57)

Division by the square of the natural circular frequency:

\[ \dddot{x} + 2\delta \dot{x} + x = 0 \]  \hspace{1cm} (4.58)

or

\[ \dddot{x} + 2D \dot{x} + x = 0 \]  \hspace{1cm} (4.59)

The parameter \( D \) is the only quantity which describes the behavior of the free damped oscillator. It is called the dimensionless damping (in Germany also called Lehr’s damping):

\[ D = \frac{\delta}{\omega_0} = \frac{c}{2m\omega_0} = \frac{c}{2\sqrt{km}} \]  \hspace{1cm} (4.60)

As we can see, \( D \) contains all three parameters \( m, c \) and \( k \).
4.3.2 Solution of the Equation of Motion

The solution to eq. (4.59) can be found by the assumption that the solution has the form:

\[ x(\tau) = A e^{\lambda \tau} \]  
\[ \lambda = \lambda_1, \lambda_2 \]  

where \( A \) and \( \lambda \) are in general complex. Putting this into eq. (4.59) we get the characteristic equation

\[ \lambda^2 + 2D\lambda + 1 = 0 \]  

which has two solutions:

\[ \lambda_{1,2} = -D \pm \sqrt{D^2 - 1} = -D \pm i\sqrt{1 - D^2} \]  

Eq. (4.63) shows that the oscillator changes its behaviour depending whether \( D \) leads to real or complex solutions of \( \lambda \) (see Fig. 4.8). The general mathematical solution for \( x(\tau) \) resp. \( x(t) \) for \( \lambda_1 \neq \lambda_2 \) is:

\[ x(\tau) = A_1 e^{\lambda_1 \tau} + A_2 e^{\lambda_2 \tau} \]  

and if \( \lambda_1 = \lambda_2 = \lambda \):

\[ x(\tau) = A_1 e^{\lambda \tau} + \tau A_2 e^{\lambda \tau} = (A_1 + \tau A_2) e^{\lambda \tau} \]
The constants \(A_{1,2}\) are determined from the initial conditions (initial displacement and velocity, respectively). They can be real or complex.

For different \(D\) we get different types of solutions, which we will discuss in the following sections.

1. **Overdamped Case: Strong Damping:**
   \(D > 1\) represents strong damping, we do not get a vibration, but the mass creeps back to the equilibrium position. The mathematical solution for this case is:
   \[
   \lambda_{1,2} = -D \pm \sqrt{D^2 - 1}
   \]
   we obtain two real (and different) values. The solution is:
   \[
   x(\tau) = A_1 e^{(-D + \sqrt{D^2 - 1})\tau} + A_2 e^{(-D - \sqrt{D^2 - 1})\tau}
   \]
   or, if we go back to the \(t\)-time scale:
   \[
   x(t) = A_1 e^{(-D + \sqrt{D^2 - 1})\omega_0 t} + A_2 e^{(-D - \sqrt{D^2 - 1})\omega_0 t}
   \]
   Because the real values of \(\lambda\) are always negative we find the creep motion back to the static equilibrium.

2. **Critical Damping:**
   \(D = 1\)
   For \(D = 1\) we get the transition from vibration to creep motion (hence we call this state critical). The two solutions of \(\lambda\) are identical and real which follows immediately from eq. (4.63):
   \[
   \lambda_{1,2} = -D \pm \sqrt{D^2 - 1} = -1 = \lambda
   \]
   The resulting motion is
   \[
   x(\tau) = A_1 e^{-\tau + \tau A_2 e^{-\tau}} = (A_1 + \tau A_2) e^{-\tau}
   \]
   and on the \(t\)-time scale:
   \[
   x(t) = A_1 e^{-\omega_0 t} + \omega_0 t A_2 e^{-\omega_0 t} = (A_1 + \omega_0 t A_2) e^{-\omega_0 t}
   \]

3. **Weak Damping, Damped Vibrations:**
   \(0 < D < 1\)
   For \(0 < D < 1\) we get:
   \[
   \lambda_{1,2} = -D \pm i\sqrt{1 - D^2}
   \]
   which are conjugate complex. The solution is:
   \[
   x(\tau) = A_1 e^{(-D + i\sqrt{1 - D^2})\tau} + A_2 e^{(-D - i\sqrt{1 - D^2})\tau}
   \]
   On the \(t\)-time scale we obtain:
   \[
   x(t) = A_1 e^{(-D + i\sqrt{1 - D^2})\omega_0 t} + A_2 e^{(-D - i\sqrt{1 - D^2})\omega_0 t}
   \]
Figure 4.9: Free vibrations of a viscously damped oscillator for different $D$
If we separate the real- and imaginary parts in the exponential function (eq. (4.71)) we get:

\[ x(t) = e^{-D\omega_0 t} \left( A_1 e^{i\sqrt{1-D^2}\omega_0 t} + A_2 e^{-i\sqrt{1-D^2}\omega_0 t} \right) \]  (4.72)

The exponential term standing left of the bracket describes the decaying behavior of the vibration (for \( D = 0 \) we obtain the undamped case without reduction of the amplitudes).

The expression with the imaginary exponent show us the oscillation because

\[ e^{\pm \omega_D t} = \cos \omega_D t \pm i\sin \omega_D t \]

and using eq. (4.72) the circular frequency of the damped oscillation is:

\[ \omega_D = \sqrt{1-D^2\omega_0} \]  (4.73)

The frequency is reduced by the influence of the damping. However, if the values for \( D \) are in the range 0.01 - 0.1, the change of the frequency compared to the undamped case is negligible.

The period (from one maximum to the next maximum) then is:

\[ T = \frac{2\pi}{\omega_D} \]  (4.74)

Using trigonometric functions instead of the exponential function eq. (4.72) becomes:

\[ x(t) = e^{-D\omega_0 t} \left[ A \cos \left( \sqrt{1-D^2}\omega_0 t \right) + B \sin \left( \sqrt{1-D^2}\omega_0 t \right) \right] \]  (4.75)

Introducing the initial displacement \( x_0 \) and initial velocity \( v_0 \) at time instant \( t = 0 \) yields the constants:

\[ A = x_0 \]  (4.76)

and

\[ B = \frac{v_0 + D\omega_0 x_0}{\omega_0 \sqrt{1-D^2}} \]  (4.77)

or alternatively

\[ x(t) = Ce^{-D\omega_0 t} \sin (\omega_D t + \phi) \]  (4.78)

The maximum amplitude \( C \) and the phase angle \( \phi \) can be determined from the following equations:

\[ C = \sqrt{A^2 + B^2} = \frac{\sqrt{(x_0 \omega_D)^2 + (v_0 + D\omega_0 x_0)^2}}{\omega_D} \]  (4.79)
and

\[ \tan \varphi = \frac{A}{B} \quad (4.80) \]

An important quantity to describe the damping behavior is the *logarithmic decrement*:

\[
\theta = \frac{1}{n} \ln \frac{\hat{x}_1}{\hat{x}_{n+1}} = 2\pi \frac{D}{\sqrt{1 - D^2}} \quad (4.81)
\]

The \( \hat{x}_i \) are maxima of the damped vibration and can be determined from measured curves of the free damped oscillations of a vibrating system. Because the use of two subsequent maxima can lead to inaccurate results we observe the decaying process over a longer time (e.g. \( n = 5 - 10 \) periods, if the damping is small enough). From the logarithmic decrement we can immediately determine \( D \). From the \( nT \) on the time axis from maximum \( \hat{x}_i \) to the maximum \( \hat{x}_{n+1} \) we can calculate the damped natural circular frequency \( \omega_D \).

### 4.4 Forced Vibrations from Harmonic Excitation

As discussed earlier, forced vibrations are one very important practical mechanism for the occurrence of vibrations. The equation of motion of the damped linear SDOF oscillator with an external force (see fig. 4.10) is:

\[ m\ddot{x} + c\dot{x} + kx = F(t) \quad (4.82) \]

The general solution of this differential equation is:

\[ x(t) = x_{\text{hom}}(t) + x_{\text{part}}(t) \quad (4.83) \]

\[ x_{\text{hom}}(t) \text{ results from external force} \]

\[ x_{\text{part}}(t) \text{ results from external force} \]

Figure 4.10: SDOF oscillator with viscous damping and external force
which consists of the homogeneous part resulting from the free vibration and
the particular part resulting from the external disturbance $F(t)$ (see fig. 4.11).
The homogeneous solution has already been treated in the last chapter.

While the homogeneous part of the solution will decay to zero with time we
are especially interested in the stationary solution.

4.4.1 Excitation with Constant Force Amplitude

The excitation function is harmonic, $\Omega$ is the frequency of excitation:

$$F(t) = \hat{F} \cos \Omega t$$

(4.84)

Eq. 4.82 becomes:

$$m\ddot{x} + c\dot{x} + kx = \hat{F} \cos \Omega t$$

(4.85)

Dividing by the mass $m$:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{\hat{F}}{m} \cos \Omega t.$$  

(4.86)

Introducing again the dimension less damping (eq. (4.60)) and the natural
circular frequency (eq. (4.4)):

$$2D = \frac{c}{m\omega_0},$$

(4.87)

and

$$\omega_0^2 = \frac{k}{m}.$$  

(4.88)

and the amplitude

$$\hat{f} = \frac{\hat{F}}{m}.$$  

(4.89)
This yields:

\[
\ddot{x} + 2D\omega_0 \dot{x} + \omega_0^2 x = \hat{f} \cos \Omega t .
\] (4.90)

To solve this differential equation, we make an approach with harmonic functions

\[
x(t) = A \cos \Omega t + B \sin \Omega t
\] (4.91)

This covers also a possible phase lag due to the damping in the system. Differentiating eq. (4.91) to get the velocity and the acceleration and putting this into eq. (4.90) leads to:

\[
-\Omega^2 A \cos \Omega t - \Omega^2 B \sin \Omega t + \ldots
\]
\[
\ldots + 2D\omega_0 \left( -\Omega A \sin \Omega t + \Omega B \cos \Omega t \right) + \ldots
\]
\[
\ldots + \omega_0^2 \left( A \cos \Omega t + B \sin \Omega t \right) = \hat{f} \cos \Omega t
\] (4.92)

After separating the coefficients of the sin- and cos-functions and comparing the coefficients we get:

\[
-\Omega^2 A + 2D\omega_0 \Omega B + \omega_0^2 A = \hat{f}
\] (4.93)
\[
-\Omega^2 B - 2D\omega_0 \Omega A + \omega_0^2 B = 0
\] (4.94)

From the second equation we see that

\[
-\Omega^2 B + \omega_0^2 B = 2D\omega_0 \Omega A,
\] (4.95)

which leads to

\[
B = \frac{2D\omega_0 \Omega}{\omega_0^2 - \Omega^2} A
\] (4.96)

and we put this result into eq. (4.93):

\[
-\Omega^2 A + 2D\omega_0 \Omega \frac{2D\omega_0 \Omega}{\omega_0^2 - \Omega^2} A + \omega_0^2 A = \hat{f}
\] (4.97)
\[
\left[ \left( \omega_0^2 - \Omega^2 \right) + \frac{4D^2 \omega_0^2 \Omega^2}{\omega_0^2 - \Omega^2} \right] A = \hat{f}
\] (4.98)
\[
\left[ \left( \omega_0^2 - \Omega^2 \right)^2 + 4D^2 \omega_0^2 \Omega^2 \right] A = \hat{f} \left( \omega_0^2 - \Omega^2 \right)
\] (4.99)

This yields the solution for \(A\) and \(B\):

\[
A = \frac{\hat{f} \left( \omega_0^2 - \Omega^2 \right)}{\left( \omega_0^2 - \Omega^2 \right)^2 + 4D^2 \omega_0^2 \Omega^2}
\] (4.100)

and

\[
B = \frac{\hat{f} \left( 2D\omega_0 \Omega \right)}{\left( \omega_0^2 - \Omega^2 \right)^2 + 4D^2 \omega_0^2 \Omega^2}
\] (4.101)
Introducing the *dimensionless ratio of frequencies*:

\[ \eta = \frac{\text{Excitation frequenz}}{\text{Natural frequenz}} = \frac{\Omega}{\omega_0} \quad (4.102) \]

leads to:

\[ A = \frac{\left( \frac{f}{\omega_0} \right) (1 - \eta^2)}{(1 - \eta^2)^2 + 4D^2\eta^2} \quad (4.103) \]

and

\[ B = \frac{\left( \frac{f}{\omega_0} \right) (2D\eta)}{(1 - \eta^2)^2 + 4D^2\eta^2} \quad (4.104) \]

with \( A \) and \( B \) we have found the solution for \( x(t) = A \cos \Omega t + B \sin \Omega t \).

Another possibility is to present the solution with amplitude and phase angle:

\[ x(t) = C \cos (\Omega t - \varphi) \quad (4.105) \]

The amplitude is:

\[ C = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{(1 - \eta^2)^2 + 4D^2\eta^2}} \frac{\hat{f}}{\omega_0} \quad (4.106) \]

Considering that \( \hat{f} = \frac{\hat{F}}{m} \) und \( \omega_0^2 = \frac{k}{m} \) we get:

\[ C = \frac{1}{\sqrt{(1 - \eta^2)^2 + 4D^2\eta^2}} \frac{\hat{F}}{k} \quad (4.107) \]

Introducing the *dimensionless magnification factor* \( V_1 \) which only depends on the frequency ratio and the damping \( D \):

\[ V_1(\eta, D) = \frac{1}{\sqrt{(1 - \eta^2)^2 + 4D^2\eta^2}} \quad (4.108) \]

we get the amplitude as

\[ C = V_1 \frac{\hat{F}}{k} \quad (4.109) \]

and the phase angle (using trigonometric functions similar as in 4.2.2 on p. 84):

\[ \tan \varphi = \frac{B}{A} = \frac{2D\eta}{1 - \eta^2} \quad (4.110) \]
In fig. 4.12(a) we can see that as $\eta$ approaches 1 the amplitude grows rapidly, and its value near or at the resonance is very sensitive to changes of the damping $D$. The maximum of the magnification curve for a given $D$ can be found at

$$\eta_{res} = \sqrt{1 - 2D^2} = \frac{\Omega_{res}}{\omega_0}$$  \hspace{1cm} (4.111)$$

If $D$ is very small then $\eta_{res} \approx 1$. The maximum amplitude for this $D$ then is:

$$C_{max} = V_1(\eta_{res}, D) \frac{\hat{F}}{k} = \frac{1}{2D\sqrt{1-D^2}} \frac{\hat{F}}{k}$$  \hspace{1cm} (4.112)$$

$\eta \to 0 \quad V_1 \approx 1$ \hspace{0.5cm} The system behaves quasi-statically.

$\eta \to \infty \quad V_1 \to 0$ \hspace{0.5cm} The vibrations are very small.
Figure 4.12: Magnification factor $V_1$ and phase angle to describe the vibration behavior of the damped oscillator under constant force amplitude excitation
4.4.2 Harmonic Force from Imbalance Excitation

The total mass of the system in fig. 4.13 consists of the mass $m_M$ and the two rotating unbalance masses $m_u$:

$$m = m_M + 2 \frac{m_u}{2} \quad (4.113)$$

The disturbance force from the unbalance is depending on the angular speed $\Omega$, $\epsilon$ the eccentricity:

$$F_{Unbalance} (t) = \Omega^2 \epsilon m_u \cos \Omega t \quad (4.114)$$

Now, following the same way as before (real or complex) leads to the solution:

$$x(t) = C \cos (\Omega t - \varphi) \quad (4.115)$$

with Amplitude

$$\hat{x} = C = \frac{\eta^2}{\sqrt{(1 - \eta^2)^2 + 4D^2\eta^2}} \epsilon \frac{m_u}{m} = V_3 (\eta, D) \epsilon \frac{m_u}{m} \quad (4.116)$$

Phase

$$\tan \varphi = \frac{2D\eta}{1 - \eta^2} \quad (4.117)$$

and Magnification factor

$$V_3 (\eta, D) = \frac{\eta^2}{\sqrt{(1 - \eta^2)^2 + 4D^2\eta^2}} \quad (4.118)$$

The phase has the same expression as in the previous case (eq. (4.110)), however, the magnification factor is different (see fig. 4.14), because the force amplitude is increasing with increasing angular speed.
As can be seen, for:
\[ \eta \to 0 \quad V_3 \approx 0 \quad \text{There is no force if the system is not rotating or}
\quad \text{rotates only slowly} \]
\[ \eta \to \infty \quad V_3 \to 1 \quad \text{The Mass } m_M \text{ is vibrating with an amplitude}
\quad (\varepsilon M), \text{ but the common center of gravity of total}
\quad \text{system } m_M \text{ and } m_u \text{ does not move.} \]

4.4.3 Support Motion / Ground Motion

4.4.3.1 Case 1

The equation of motion for this system depicted in fig. 4.15 is:
\[ m\ddot{x} + c\dot{x} + kx = ku(t) \] (4.119)

Under harmonic excitation:
\[ u(t) = \hat{u} \cos \Omega t \] (4.120)
The mathematical treatment is nearly identical to the first case, only the excitation function is different: the excitation $\hat{F}_k$ is replaced by $\hat{u}$ here. This leads to the result for the amplitude of vibration:

$$\hat{x} = \frac{1}{\sqrt{(1 - \eta^2)^2 + 4D^2\eta^2}} \hat{u} = V_1(\eta, D) \hat{u}$$  \hspace{1cm} (4.121)

The magnification factor again is $V_1$ (eq. (4.108)). Also, the phase relation is identical as before (eq. (4.110)):

$$\tan \varphi = \frac{2D\eta}{1 - \eta^2}$$  \hspace{1cm} (4.122)

### 4.4.3.2 Case 2

The equation of motion now (see fig. 4.16) also contains the velocity $\dot{u}$:

$$m\ddot{x} + c\dot{x} + kx = c\dot{u} + ku$$  \hspace{1cm} (4.123)

Amplitude of vibration and phase shift become:

$$\hat{x} = \frac{\sqrt{1 + 4D^2\eta^2}}{\sqrt{(1 - \eta^2)^2 + 4D^2\eta^2}} \hat{u} = V_2(\eta, D) \hat{u}$$  \hspace{1cm} (4.124)

and

$$\tan \varphi = \frac{2D\eta^3}{1 - \eta^2 + 4D^2\eta^2}.$$  \hspace{1cm} (4.125)
As can be seen the phase now is different due to the fact that the damper force depending on the relative velocity between ground motion and motion of the mass plays a role. The amplitude behaviour is described by the magnification factor:

\[ V_2 = \frac{\sqrt{1 + 4D^2\eta^2}}{\sqrt{(1 - \eta^2)^2 + 4D^2\eta^2}} \]  

(4.126)

Notice that all curves have an intersection point at \( \eta = \sqrt{2} \) (see fig. 4.17 which means that for \( \eta > \sqrt{2} \) higher damping does not lead to smaller amplitudes but increases the amplitudes. This is due to the fact that larger relative velocities (due to higher frequencies \( \eta \)) make the damper stiffer and hence the damping forces.

Further cases of ground motion excitation are possible.

4.5 EXCITATION BY IMPACTS

4.5.1 Impact of Finite Duration

We consider an impact of finite length \( T_i \) and constant force level during the impact (see fig. 4.18). The impact duration \( T_i \) is much smaller than the period of vibration \( T \).

\[ T_i << T = \frac{2\pi}{\omega_D} \]  

(4.127)
With the initial condition that there is no initial displacement \( x_0 = 0 \) we can calculate the velocity by means of the impulse of the force:

\[
p = mv_0 = \int_0^{T_i} \hat{F} \, dt = \hat{F} T_i
\]  

(4.128)

This leads to the initial velocity:

\[
v_0 = \frac{\hat{F} T_i}{m}
\]  

(4.129)

---

**Figure 4.17:** Magnification factor \( V_2 \) for the case of ground excitation via spring and damper

**Figure 4.18:** SDOF oscillator under impact loading
Using the results of the viscously damped free oscillator for $D < 1$ (eq. (4.75)),

$$x(t) = e^{-D\omega_0 t} \left[ A \cos \left( \frac{\sqrt{1 - D^2\omega_0^2} t}{\omega_D} \right) + B \sin \left( \frac{\sqrt{1 - D^2\omega_0^2} t}{\omega_D} \right) \right]$$

we can immediately find the result with the initial conditions $x_0$ und $v_0$:

$$x_0 = 0 \quad \Rightarrow \quad A = 0 \quad (4.131)$$

and

$$v_0 = \frac{\hat{F}T_i}{m} \quad \Rightarrow \quad B = \frac{v_0}{\omega_0 \sqrt{1 - D^2}} = \frac{v_0}{\omega_D} \quad (4.132)$$

so that the system response to the impact is a decaying oscillation where we have assumed that the damping $D < 1$:

$$x(t) = \frac{v_0}{\omega_D} e^{-D\omega_0 t} \sin(\omega_D t)$$

(4.133)

4.5.2 DIRAC-Impact

The DIRAC-Impact is defined by

$$F(t) = \hat{F}\delta(t) \rightarrow \delta(t) = \begin{cases} 0 \quad t \neq 0 \\ \infty \quad t = 0 \end{cases} \quad \text{but} \quad \int_{-\infty}^{\infty} \delta(t) \, dt = 1$$

(4.134)

$\delta$ is the Kronecker symbol. The duration of this impact is infinitely short but the impact is infinitely large. However, the integral is equal to 1 or $\hat{F}$,
respectively. For the initial displacement $x_0 = 0$ and calculation of the initial velocity following the previous chapter, we get:

$$x(t) = \frac{\hat{F}}{m\omega_D} e^{-D\omega_D t} \sin(\omega_D t)$$  \hspace{1cm} (4.135)$$

For $\hat{F} = 1$, the response $x(t)$ is equal to the impulse response function (IRF) $h(t)$:

$$h(t) = \frac{1}{m\omega_D} e^{-D\omega_D t} \sin(\omega_D t)$$  \hspace{1cm} (4.136)$$

The IRF is an important characteristic of a dynamic system in control theory.

4.6 EXCITATION BY FORCES WITH ARBITRARY TIME FUNCTIONS

![Figure 4.20](image_url)

Figure 4.20: Interpretation of an arbitrary time function as series of DIRAC-impacts

Using the results of the previous chapters we can solve the problem of an arbitrary time function $F(t)$ (see fig. 4.20) as subsequent series of Dirac-impacts, where the initial conditions follow from the time history of the system.
The solution is given by the Duhamel-Integral or convolution integral:

$$x(t) = \int_0^t \frac{1}{m\omega_D} e^{-D\omega_0 (t-\tau)} \sin(\omega_D (t-\tau)) F(\tau) \, d\tau \quad (4.137)$$

$$= \int_0^t h(t-\tau) F(\tau) \, d\tau \quad (4.138)$$

As can be seen, the integral contains the response of the SDOF oscillator with respect to a DIRAC-impact multiplied with the actual force $F(\tau)$, which is integrated from time $0$ bis $t$.

4.7 PERIODIC EXCITATIONS

4.7.1 Fourier Series Representation of Signals

Periodic signals can be decomposed into an infinite series of trigonometric functions, called Fourier series (see fig. 4.21). The period of the signal is $T$ and the corresponding fundamental frequency is:

$$\omega = \frac{2\pi}{T} \quad (4.139)$$

Now, the periodic signal $x(t)$ can be represented as follows:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t) \quad (4.140)$$

Figure 4.21: Scheme of signal decomposition by trigonometric functions
The Fourier-coefficients $a_0$, $a_k$ and $b_k$ must be determined. They describe how strong the corresponding trigonometric function is present in the signal $x(t)$. The coefficient $a_0$ is the double mean value of the signal in the interval $0 \ldots T$:

$$a_0 = \frac{2}{T} \int_0^T x(t) \, dt \quad (4.141)$$

and represents the off-set of the signal. The other coefficients can be determined from

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega t) \, dt \quad (4.142)$$

and

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega t) \, dt \quad (4.143)$$

The individual frequencies of this terms are

$$\omega_k = k\omega = \frac{2\pi k}{T} \quad (4.144)$$

For $k = 1$ we call the frequency $\omega_1$ fundamental frequency or basic harmonic and the frequencies for $k = 2, 3, \ldots$ the second, third, ... harmonic (or generally higher harmonics).

The real trigonometric functions can also be transformed into complex exponential expression:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega t} \quad (4.145)$$

The $X_k$ are the complex Fourier coefficients which can be determined by solving the integral:

$$X_k = \frac{1}{T} \int_0^T x(t) e^{-ik\omega t} \, dt \quad (4.146)$$

or

$$X_k = \frac{1}{T} \int_0^T x(t) \left[\cos k\omega t - i \sin k\omega t\right] \, dt \quad (4.147)$$

which clearly shows the relation to the real Fourier coefficients series given by eq. (4.142) and eq. (4.143):

$$\text{Re} \{X_k\} = \frac{a_k}{2}; \quad \text{Im} \{X_k\} = -\frac{b_k}{2} \quad (4.148)$$

The connection to the other real representation (chap. 4.7.1) is:

$$|X_k| = c_k \quad \tan \varphi_k = \left(\frac{\text{Im} \{X_k\}}{\text{Re} \{X_k\}}\right) \quad (4.149)$$

The coefficients with negative index are the conjugate complex values of the corresponding positive ones:

$$X_{-k} = X_k^* \quad (4.150)$$
4.7.2 *Forced Vibration Under General Periodic Excitation*

Let us use once more the SDOF oscillator (see fig. 4.22) but now the force is a periodic function which can be represented by a *Fourier* series

\[
F(t) = \frac{F_0}{2} + \sum_{k=1}^{\infty} F_{ck} \cos (k\Omega t) + F_{sk} \sin (k\Omega t)
\]  

(4.151)

The \(F_{ck}\) and \(F_{sk}\) are the *Fourier* coefficients which can be determined according to the last chapter (eqns. 4.141 - 4.143). The response due to such an excitation is:

\[
x(t) = \frac{F_0}{2k^*} + \sum_{k=1}^{\infty} V_1(\eta_k, D) \frac{F_{ck}}{k^*} \cos (k\Omega t - \varphi_k) + \ldots
\]

\[
\ldots + V_1(\eta_k, D) \frac{F_{sk}}{k^*} \sin (k\Omega t - \varphi_k)
\]

(4.152)

with the frequency ratio:

\[
\eta_k = \frac{k\Omega}{\omega_0}, \; k = 1, 2, \ldots \infty
\]

(4.153)

Each individual frequency is considered with its special amplification factor \(V\) and individual phase shift \(\varphi\), which in the present case can be calculated from (see eq. (4.108)):

\[
V_1(\eta_k, D) = \frac{1}{\sqrt{(1 - \eta_k^2)^2 + 4D^2\eta_k^2}}
\]

\[
\tan \varphi_k = \frac{2D\eta_k}{1 - \eta_k^2}
\]

For the other cases of mass unbalance excitation or ground excitation the procedure works analogously. The appropriate \(V\)-functions have to be used and the correct pre-factors (which is in the present case \(\frac{1}{k}\)) have to be used.
EXAMPLE: Graphical determination of the complete solution for a given perturbation function \( F(t) \) (see fig. 4.23)

\[
F(t) = F_1(t) + F_3(t) = \hat{F}_1 \sin(\Omega_1 t) + \hat{F}_3 \sin(\Omega_3 t)
\]

Fig. 4.24 shows the graphical representation of both partial functions of the perturbation function (only the first two not vanishing parts of the Fourier series) where:

\[
F_1(t) = \hat{F}_1 \sin(\Omega_1 t) \quad F_3(t) = \hat{F}_3 \sin(\Omega_3 t)
\]

\[
\hat{F}_1 = 1 \quad \hat{F}_3 = \frac{1}{3} \hat{F}_1
\]

\[
\Omega_1 = \Omega \quad \Omega_3 = 3\Omega
\]

The frequency-dependent amplitude (magnification factor) including the values \( V_1 \) and \( V_3 \) which are determined for \( \eta_1 \) and \( \eta_3 \) is depicted in a resonance diagram shown in fig. 4.25(a) (dimensionless damping amounts \( D = 0.2 \)). Fig. 4.25(b) shows the according phase angle diagram. The graphical representations of the partial solutions

\[
x_1(t) = A_1 \sin(\Omega_1 t - \varphi_1)
\]

\[
x_3(t) = A_3 \sin(\Omega_3 t - \varphi_3)
\]

are illustrated in fig. 4.25(c). In fig. 4.25(d) one can study the complete solution

\[
x(t) = x_1(t) + x_3(t)
\]

with \( A_1 > A_3 \).
The difference in case of resonance is shown by the graphical representation of the frequency-dependent amplitude (magnification factor) including the values

![Graphical representation of both sub-functions](image)

**Figure 4.24:** Graphical representation of both sub-functions

(a) Magnification factor

(b) Phase angle

(c) Partial solution

(d) Complete solution

![Graphical determination of the complete solution](image)

**Figure 4.25:** Graphical determination of the complete solution for a given perturbation function $F(t)$ without resonance
4.7 Periodic Excitations

Figure 4.26: Graphical determination of the complete solution for a given perturbation function \( F(t) \) in case of resonance

\( V_1 \) and \( V_3 \) which are determined for \( \eta_1 \) and \( \eta_3 \). This is depicted in a resonance diagram in fig. 4.26(a). Here we have resonance, dimensionless damping amounts \( D = 0.2 \). Fig. 4.26(b) shows the according phase angle diagram. The graphical representations of the partial solutions

\[
x_1(t) = A_1 \sin (\Omega_1 t - \varphi_1)
\]

\[
x_3(t) = A_3 \sin (\Omega_3 t - \varphi_3)
\]

are illustrated in fig. 4.26(c). In fig. 4.26(d) one can study the complete solution

\[
x(t) = x_1(t) + x_3(t)
\]

with \( A_1 < A_3 \).
4.8 VIBRATION ISOLATION OF MACHINES

In dynamics of machines and systems the linear vibrations of SDOF systems introduced at the beginning of chapter 4 are often applied to isolate machinery from vibrations.

- **Requirements:**
  - machine can be idealized as rigid
  - vibration process can be considered to be a SDOF system

- **Vibration isolation:**
  - important assignment of dynamics of machines and systems
  - concerning machine manufactures, project planers, operators
  - any installation site whether ground, floor or another carrying structure is elastic. That is why excitation by imbalanced inertia forces can occur

- **Purpose:**
  - installing the machine in order to minimize forces (inertia forces, ...) diverted into the foundation / structure
  - mounting sensitive devices in a way so that they are well shielded from their vibrating environment

We differ between:

1. **Low tuning:**
   The highest natural frequency of the ground vibration is less than the lowest excitation frequency.

2. **High tuning:**
   The natural frequencies of the ground vibration are above the range of the excitation frequencies.

3. **Mixed tuning:**
   The range of natural frequencies and the range of excitation frequencies partially overlap each other, but there is no resonance.
These terms are generally valid for MDOF systems too.

4.8.1 Forces on the Environment Due to Excitation by Inertia Forces

In order to achieve vibration isolation we initially have to calculate the forces on the environment in order to use encountered dependencies for the purpose of minimizing these forces. The determination of these forces is performed by means of an example.

**Example:** Imbalanced machine (see fig. 4.27)
Amplitude of the excitation force:

$$\hat{F}_u = m_u \epsilon \Omega^2 = U \Omega^2$$

with total mass $m$ and imbalance mass $m_u$.

Forces acting on the foundation like in fig. 4.28:

$$F_{Fu} = kx + c\dot{x}$$

In case of harmonic excitation $x(t)$:

$$\hat{X}(t) = \hat{X} e^{i\Omega t}$$
$$\dot{\hat{X}}(t) = i\Omega \hat{X} e^{i\Omega t}$$
$$\ddot{\hat{X}}(t) = -\Omega^2 \hat{X} e^{i\Omega t}$$

$$\hat{F}_{Fu}(t) = k\hat{X} e^{i\Omega t} + ci\Omega \hat{X} e^{i\Omega t} = (k + i\Omega c) \hat{X} e^{i\Omega t}$$

$$\hat{F}_{Fu}(t) = (k + i\Omega c) \frac{\eta^2}{(1 - \eta^2) + 2D\eta i} \frac{m_u}{m}$$
$$= \left(\omega_0^2 + 2D\omega_0 i\Omega\right) \frac{\eta^2}{(1 - \eta^2) + 2D\eta i} \epsilon m_u$$

Referring the complex amplitude to the amplitude of the excitation force:

$$\frac{\hat{F}_{Fu}}{\hat{F}_u} = \frac{\left(\omega_0^2 + 2D\omega_0 i\Omega\right) \frac{\eta^2}{(1 - \eta^2) + 2D\eta i} \epsilon m_u}{\Omega^2 \epsilon m_u} = \frac{1 + 2D\eta i}{1 - \eta^2 + 2D\eta i}$$

Real amplitude ratio (see. eq. (4.126)):

$$\frac{\hat{F}_{Fu}}{\hat{F}_u} = \sqrt{\frac{1 + 4D^2\eta^2}{(1 - \eta^2)^2 + 4D^2\eta^2}} = V_2 (\eta, D) \quad (4.154)$$

Absolute amplitude of the foundation force:

$$\hat{F}_{Fu} = V_2 (\eta, D) \hat{F}_u = V_2 \Omega^2 m_u \epsilon = V_2 \eta^2 m_u \epsilon \omega_0^2 = V_2 \eta^2 \frac{m_u}{m} \epsilon k \quad (4.155)$$

$$\hat{F}_{Fu} = V_2 \eta^2 \frac{m_u}{m} \epsilon k = V_4 \frac{m_u}{m} \epsilon k \quad (4.156)$$

$$V_4 = \eta^2 V_2 \quad (4.157)$$
4.8.2 Tuning of Springs and Dampers

From the real amplitude ration (see eq. (4.154))

\[
\frac{\hat{F}_F}{\hat{F}_u} = V_2(\eta, D),
\]

(4.158)

we can deduce, \(\eta\) should attain higher values so that \(V_2\) becomes as small as possible (see fig. 4.29).

High values of \(\eta\) mean:

\[
\Omega = \eta \omega_0 \quad \Rightarrow \quad \Omega > \omega_0
\]

This is called low or 'soft' tuning.

Soft springs, heavy foundation block \(\Rightarrow\) small \(\omega_0\).

Using soft springs, it is necessary to mind the static lowering under dead load, because it can increase to high values (too high!).

Moreover it shows: In order to have \(V_2\) as small as possible damping should be smallest possible at fixed \(\Omega\). But this brings along disadvantages for passing the case of resonance (while starting up and shutting down).
Remedy:

- passing resonance frequency very fast (sufficient torque required!)
- connecting a damper while passing resonance frequency

By comparison: rigid installation would lead to $\frac{\hat{F}_{Fu}}{F_u} = 1$.

Numerical example  $D = 0$; Tuning target: we want to tune the foundation in a way, so that only 5\% of the excitation force is transferred to the environment!

\[
\frac{\hat{F}_{Fu}}{F_u} < 5\% = \frac{1}{20}
\]

\[
V_2 (\eta, D = 0) = \frac{1}{1 - \eta^2}
\]

\[
\eta = \frac{\Omega}{\omega_0}
\]

\[
\Rightarrow \frac{1}{20} = \pm \left| \frac{1}{1 - \eta^2} \right|
\]

\[
\Rightarrow 1 - \eta^2 = -20
\]

\[
\Leftrightarrow \eta^2 = -21
\]

\[
\Leftrightarrow \eta = \sqrt{21} \approx 4.58
\]

that means

\[
\Omega \geq 4.58\omega_0 \text{ or } \omega_0 \leq \left( \frac{1}{4.58} \right) \Omega \quad (4.159)
\]

In case of constant force amplitudes we obtain the following isolation effect:

\[
F (t) = \hat{F} \sin \Omega t
\]

or

\[
\tilde{F} (t) = \hat{F} e^{i\Omega t}.
\]

Applying the same approach as in section 4.8 we get (see eq. (4.154)):

\[
\frac{\hat{F}_{Fu}}{F_u} = V_2 (\eta, D)
\]

Dealing with machinery installed in way that it is excited by ground motion we have (see eq. (4.126)):

\[
\frac{\hat{x}}{x_a} = V_2 (\eta, D)
\]
We notice the most suitable way to manage a vibration isolated installation of sensitive devices is to use low tuning. The same applies to soundproof rooms etc.
5.1 EQUATION OF MOTION

The equation of motion can be derived by using the principles we have learned such as Newton’s/Euler’s laws or Lagrange’s equation of motion. For a general linear system mDOF system we found that we can write in matrix form:

\[
\begin{bmatrix}
M & B & \cdots & 0 \\
B & M & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\vdots \\
\ddot{x}_m
\end{bmatrix}
+ 
\begin{bmatrix}
C & \cdots & 0 \\
\cdots & \cdots & \cdots \\
0 & \cdots & C
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_m
\end{bmatrix}
+ 
\begin{bmatrix}
K & \cdots & 0 \\
\cdots & \cdots & \cdots \\
0 & \cdots & K
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_m
\end{bmatrix}
= 
\begin{bmatrix}
F_1 \\
F_2 \\
\vdots \\
F_m
\end{bmatrix}
\]

(5.1)

with the matrices:

- \( M \): Mass matrix (symmetric) \( M = M^T \)
- \( C \): Damping matrix (symmetric) \( C = C^T \)
- \( K \): Stiffness matrix (symmetric) \( K = K^T \)
- \( G \): Gyroscopic matrix (skew-symmetric) \( G = -G^T \)
- \( N \): Matrix of non-conservative forces (skew-symmetric) \( N = -N^T \)
- \( F \): External forces

Note:
A general matrix \( B \) can be decomposed into the symmetric part and the skew-symmetric part \( \bar{B} \) and \( \tilde{B} \):

\[
B = B + \tilde{B} + \frac{1}{2} (B - B^T)
\]

where:

\[
\bar{B} = \frac{1}{2} (B + B^T)
\]

In the standard case that we have no gyroscopic forces and no non-conservative displacement dependent forces but only inertial forces, damping forces and elastic forces the last equation reduces to:

\[
M \ddot{x} + C \dot{x} + K x = F
\]

(5.2)
EXAMPLE  The system shown in fig. 5.1, where the masses can slide without friction ($\mu = 0$), has the

$$
\begin{bmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3 \\
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\ddot{x}_3 \\
\end{bmatrix}
+ 
\begin{bmatrix}
c_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\end{bmatrix}
+ \ldots
$$

$$
\begin{bmatrix}
k_1 + k_2 + k_3 & -k_2 & -k_3 \\
-k_2 & k_2 & 0 \\
-k_3 & 0 & k_3 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
F_1(t) \\
F_2(t) \\
F_3(t) \\
\end{bmatrix}
$$

(5.3)

5.2 INFLUENCE OF THE WEIGHT FORCES AND STATIC EQUILIBRIUM

The static equilibrium displacements are calculated by ($\ddot{x}_{stat} = \dot{x}_{stat} = 0$):

$$
K\ddot{x}_{stat} = F_{stat}
$$

(5.4)

which means for the case of the example shown in fig. 5.2:

$$
K\ddot{x}_{stat} = F_{stat} = \begin{pmatrix} m_1 g \\ m_2 g \end{pmatrix}
$$

(5.5)

Figure 5.1: Multi-Degree-of-Freedom (MDOF) system with free body diagram
The dynamic problem for this example is:

\[ M \ddot{x} + K x = \underbrace{F_{\text{stat}}}_{\text{static force}} + F(t) \]  

(5.6)

\[ x = \underbrace{x_{\text{stat}}}_{\text{part of the motion}} + \underbrace{x_{\text{dyn}}}_{\text{describing the vibration about the static equilibrium}} \]  

(5.7)

From the last equation also follows that

\[ \dot{x} = \dot{x}_{\text{dyn}} \Rightarrow \ddot{x} = \ddot{x}_{\text{dyn}} \]  

(5.8)

so that

\[ M \ddot{x}_{\text{dyn}} + K (x_{\text{dyn}} + x_{\text{stat}}) = F_{\text{stat}} + F(t) \]  

(5.9)

and after rearrangement

\[ M \ddot{x}_{\text{dyn}} + K x_{\text{dyn}} = F_{\text{stat}} - K x_{\text{stat}} + F(t) \]  

(5.10)

\[ M \ddot{x}_{\text{dyn}} + K x_{\text{dyn}} = F(t) \]  

(5.11)

As can be seen the static forces and static displacements can be eliminated and the equation of motion describes the dynamic process about the static equilibrium position.

In cases where the weight forces influences the dynamic behavior a simple elimination of the static forces and displacements is not possible. In the example of an inverted pendulum shown in fig. 5.3 the restoring moment is \( mg l \sin \phi \), where \( l \) is the length of the pendulum.
5.3 GROUND EXCITATION

Fig. 5.4 shows a MDOF system.

Without ground motion \( x_0 = 0 \) the equation of motion is:

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2 \\
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2 \\
\end{bmatrix}
+ \begin{bmatrix}
  c_1 & 0 \\
  0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2 \\
\end{bmatrix}
+ \ldots

\ldots + \begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2 \\
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
\end{bmatrix}
= \begin{bmatrix}
  F_1 \\
  F_2 \\
\end{bmatrix} \tag{5.12}
\]

Now, if we include the ground motion, the differences \( (x_1 - x_0) \) and the relative velocity \( d(x_1 - x_0)/dt \) determine the elastic and the damping force, re-
pectively at the lower mass. This can be expressed by adding $x_0$ to the last equation in the following manner:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \ldots$$

$$\ldots = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} k_1 \\ 0 \end{bmatrix} x_0(t) + \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \dot{x}_0(t)$$

(5.13)

The dynamic force $f_0$ of the vibrating system on the foundation is:

$$F_0 = k_1 (x_1 - x_0) + c_1 (\dot{x}_1 - \dot{x}_0)$$

(5.14)

Figure 5.5: Example for ground motion excitation of a building structure (earthquake excitation)

Figure 5.6: Excitation of a vehicle by rough surface
5.4 Free Undamped Vibrations of the Multiple-Degree-of-Freedom System

5.4.1 Eigensolution, Natural Frequencies and Mode Shapes of the System

The equation of motion of the undamped system is:

\[
\begin{bmatrix}
M & 0 \\
0 & K
\end{bmatrix} \ddot{x} + \begin{bmatrix}
K & 0 \\
0 & M
\end{bmatrix} x = 0
\] (5.15)

To find the solution of the homogeneous differential equation, we make the harmonic solution approach as in the SDOF case. However, now we have to consider a distribution of the individual amplitudes for each coordinate. This is done by introducing (the unknown) vector \( \varphi \):

\[
x = \varphi e^{i\omega t}
\] (5.16)

\[
\ddot{x} = -\omega^2 \varphi e^{i\omega t}
\] (5.17)

Putting this into eq. (5.15) yields:

\[
\begin{bmatrix}
K - \omega^2 M
\end{bmatrix} \varphi = 0
\] (5.18)

This is a homogeneous equation with unknown scalar \( \omega \) and vector \( \varphi \). If we set \( \lambda = \omega^2 \) we see that this is a general matrix eigenvalue problem:\footnote{The well-known special eigenvalue problem has the form \((A - \lambda I)x = 0\), where \(I\) is the identity matrix, \(x\) the eigenvector and \(\lambda\) the eigenvalue.}:

\[
\begin{bmatrix}
K - \lambda M
\end{bmatrix} \varphi = 0
\] (5.19)

where \( \lambda \) is the eigenvalue and \( \varphi \) is the eigenvector. Because the dimension of the matrices is \( f \) by \( f \) we get \( f \) pairs of eigenvalues and eigenvectors:

\[
\lambda_i = \omega_i^2 \quad \text{for} \quad i = 1, 2, \ldots, f
\] (5.20)

\( \omega_i \) is the \( i \)-th natural circular frequency and \( \varphi_i \) the \( i \)-th eigenvector which has the physical meaning of a vibration mode shape.

The solution of the characteristic equation

\[
\det \left( K - \omega^2 M \right) = 0
\] (5.21)
yields the eigenvalues and natural circular frequencies $\lambda = \omega^2$, respectively. The natural frequencies are:

$$f_i = \frac{\omega_i}{2\pi} \quad (5.22)$$

The natural frequencies are the resonant frequencies of the structure.

The eigenvectors can be normalized arbitrarily, because they only represent a vibration mode shape, no absolute values. Commonly used normalizations are

- normalize $\varphi_i$ so that $|\varphi_i| = 1$
- normalize $\varphi_i$ so that the maximum component is 1
- normalize $\varphi_i$ so that the modal mass (the generalized mass) is 1.

**Generalized mass or modal mass:**

$$M_i = \varphi_i^T M \varphi_i \quad (5.23)$$

**Generalized stiffness:**

$$K_i = \varphi_i^T K \varphi_i \quad (5.24)$$

where

$$K_i = \varphi_i^T K \varphi_i = \omega_i^2 \quad \text{if} \quad M_i = 1 \quad (5.25)$$

The so-called Rayleigh ratio is:

$$\omega_i^2 = \frac{K_i}{M_i} = \frac{\varphi_i^T K \varphi_i}{\varphi_i^T M \varphi_i} \quad (5.26)$$

It allows the calculation of the frequency if the vectors are already known.

### 5.4.2 Modal Matrix, Orthogonality of the Mode Shape Vectors

If we order the natural frequencies so that

$$\omega_1 \leq \omega_2 \leq \omega_3 \leq \ldots \leq \omega_f$$

and put the corresponding eigenvectors columnwise in a matrix, the so-called modal matrix, we get:

$$\Phi = [\varphi_1, \varphi_2, \ldots, \varphi_f] = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \ldots & \varphi_{1f} \\ \varphi_{21} & \varphi_{22} & \ldots & \varphi_{2f} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{f1} & \varphi_{f2} & \ldots & \varphi_{ff} \end{bmatrix} \quad (5.27)$$
The first subscript of the matrix elements denotes the number of the vector component while the second subscript characterizes the number of the eigenvector.

The eigenvectors are linearly independent and moreover they are orthogonal. This can be shown by a pair $i$ and $j$

$$\left( K - \omega_i^2 M \right) \varphi_j = 0 \quad \text{and} \quad \left( K - \omega_j^2 M \right) \varphi_i = 0 \quad (5.28)$$

Premultiplying by the transposed eigenvector with index $j$ and $i$ respectively:

$$\varphi_j^T \left( K - \omega_i^2 M \right) \varphi_i = 0 \quad \text{and} \quad \varphi_i^T \left( K - \omega_j^2 M \right) \varphi_j = 0 \quad (5.29)$$

If we take the transpose of the second equation

$$\varphi_j^T \left( K^T - \omega_j^2 M^T \right) \varphi_i = 0 \quad (5.30)$$

and consider the symmetry of the matrices: $M = M^T$ and $K = K^T$ and subtract this equation $\varphi_j^T \left( K - \omega_i^2 M \right) \varphi_i = 0$ from the first eq. (5.28) we get

$$\left( \omega_j^2 - \omega_i^2 \right) \varphi_j^T M \varphi_i = 0 \quad (5.31)$$

which means that if the eigenvalues are distinct $\omega_i \neq \omega_j$ for $i \neq j$ the second scalar product expression must be equal to zero:

$$\varphi_j^T M \varphi_i = 0 \quad (5.32)$$

That means that the two distinct eigenvectors $i \neq j$ are orthogonal with respect to the mass matrix. For all combinations we can write:

$$\begin{cases} 
\varphi_j^T M \varphi_i = \delta_{ij} M_i \\
\varphi_j^T K \varphi_i = \delta_{ij} \omega_i^2 M_i 
\end{cases} \quad \delta_{ij} = \begin{cases} 
0, & \text{for } i = j \\
1, & \text{for } i \neq j 
\end{cases} \quad (5.33)
$$

or with the modal matrix:

$$\Phi^T M \Phi = diag \{ M_i \} = \begin{bmatrix} 
M_1 & 0 & \ldots & 0 \\
0 & M_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & M_f 
\end{bmatrix} \quad (5.34)$$

$$\Phi^T K \Phi = diag \{ \omega_i^2 M_i \} = \begin{bmatrix} 
\omega_1^2 M_1 & 0 & \ldots & 0 \\
0 & \omega_2^2 M_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & \omega_f^2 M_f 
\end{bmatrix} \quad (5.34)$$
Figure 5.7: Mode shapes and natural frequencies of a two storey structure
5.4.3 Free Vibrations, Initial Conditions

The free motion of the undamped system $x(t)$ is a superposition of the modes vibrating with the corresponding natural frequency:

$$x(t) = \sum_{i=1}^{f} \phi_i (A_{ci} \cos \omega_i t + A_{si} \sin \omega_i t)$$  \hspace{1cm} (5.35)

Each mode is weighted by a coefficient $A_{ci}$ and $A_{si}$ which depend on the initial displacement shape and the velocities. In order to get these coefficients, we premultiply eq. (5.35) by the transposed $j$-th eigenvector:

$$\phi_j^T M x(t) = \sum_{i=1}^{f} \phi_j^T M \phi_i (A_{ci} \cos \omega_i t + A_{si} \sin \omega_i t)$$

for $i \neq j \Rightarrow = 0$

$$= \phi_j^T M \phi_j (A_{cj} \cos \omega_j t + A_{sj} \sin \omega_j t)$$  \hspace{1cm} (5.36)

All but one of the summation terms are equal to zero due to the orthogonality conditions. With the initial conditions for $t = 0$ we can derive the coefficients:

$$\begin{bmatrix} t = 0 \\ x(t = 0) = x_0 \\ \phi_j^T M x_0 = M_j A_{cj} \end{bmatrix} \Rightarrow \begin{bmatrix} A_{cj} = \frac{\phi_j^T M x_0}{M_j} \end{bmatrix}$$  \hspace{1cm} (5.38)

$$\dot{x}(t) = \sum_{i=1}^{f} \phi_i' \omega_i (-A_{ci} \sin \omega_i t + A_{si} \cos \omega_i t)$$  \hspace{1cm} (5.39)

$$\begin{bmatrix} t = 0 \\ \dot{x}(t = 0) = v_0 \\ \phi_j^T M \dot{x}_0 = M_j \omega_j A_{sj} \end{bmatrix} \Rightarrow \begin{bmatrix} A_{sj} = \frac{\phi_j^T M v_0}{M_j \omega_j} \end{bmatrix}$$  \hspace{1cm} (5.40)

which we have to calculate for modes $j$.

5.4.4 Rigid Body Modes

As learned earlier the constraints reduce the DOFs of the rigid body motion. If the number of constraints is not sufficient to suppress rigid body motion the system has also zero eigenvalues. The number of zero eigenvalues corresponds directly to the number of rigid body modes. In the example shown in fig. 5.8 the
two masses which are connected with a spring can move with a fixed distance as a rigid system. This mode is the rigid body mode, while the vibration of the two masses is a deformation mode. The equation of motion of this system is:

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
  k & -k \\
  -k & k
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

The corresponding eigenvalue problem is:

\[
\begin{bmatrix}
  k - \lambda m_1 & -k \\
  -k & k - \lambda m_2
\end{bmatrix}
\begin{bmatrix}
  \varphi_1 \\
  \varphi_2
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

The eigenvalues follow from the determinant which is set equal to zero:

\[
\begin{aligned}
\det \left[ \begin{array}{cc}
(k - \lambda m_1)(k - \lambda m_2) - k^2 &= 0 \\
\lambda^2 m_1 m_2 - \lambda (km_1 + km_2) &= 0
\end{array} \right]
\end{aligned}
\]

Obviously, this quadratic equation has the solution

\[
\lambda_1 = \omega_1^2 = 0
\]

and

\[
\lambda_2 = \omega_2^2 = k \frac{m_1 + m_2}{m_1 m_2}
\]

The corresponding (unnormalized) eigenvectors are

\[
\varphi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

which is the rigid body mode: both masses have the same displacement, no potential energy is stored in the spring and hence no vibration occurs. The second eigenvector, the deformation mode is

\[
\varphi_2 = \begin{bmatrix} 1 \\ - \frac{m_1}{m_2} \end{bmatrix}
\]

which is a vibration of the two masses. Other examples for systems with rigid body modes are shown in the following figures.

Figure 5.8: A two-DOFs oscillator which can perform rigid body motion
5.5 Forced Vibrations of the Undamped Oscillator Under Harmonic Excitation

The equation of motion for the type of system depicted in fig. 5.12 is:

\[ M \ddot{x} + K x = F(t) \]  

(5.49)

Figure 5.9: Examples for systems with torsional and transverse bending motion with rigid body motion

Figure 5.10: Flying airplane (Airbus A318) as a system with 6 rigid body modes and deformation modes

Figure 5.11: Commercial communication satellite system (EADS) with 6 rigid body modes and deformation modes
For a harmonic excitation we can make an exponential approach to solve the problem as we did with the SDOF system.

\[ F(t) = \hat{F} e^{i\Omega t} \]  (5.50)

We make a complex harmonic approach for the displacements with \( \Omega \) as the excitation frequency:

\[ x = \hat{X} e^{i\Omega t} \]  (5.51)

The acceleration vector is the second derivative

\[ \ddot{x} = -\Omega^2 \hat{X} e^{i\Omega t} \]  (5.52)

Putting both into the equation of motion (eq. (5.49)) and eliminating the exponential function yields

\[(K - \Omega^2 M) \hat{X} = \hat{F}\]  (5.53)

which is a complex linear equation system that can be solved by hand for a small number of DOFs or numerically. The formal solution is:

\[ \hat{X} = (K - \Omega^2 M)^{-1} \hat{F} \]  (5.54)

which can be solved if determinant of the coefficient matrix:

\[ \det(K - \Omega^2 M) \neq 0 \]  (5.55)

If the excitation frequency \( \Omega \) coincides with one of the natural frequencies \( \omega_i \) we get resonance of the system with infinitely large amplitudes (in the undamped case).

Figure 5.12: Example for a system under forced excitation
Resonance:
\[
\det (K - \Omega^2 M) = 0 \Leftrightarrow \Omega = \omega_i
\] (5.56)
Part II

APPENDIX
The position vector in cartesian coordinates with time invariant unit vectors \( e_x \), \( e_y \) und \( e_z \):

\[
\mathbf{r}(t) = x(t)\, e_x + y(t)\, e_y + z(t)\, e_z \quad \text{(B.1)}
\]

The coordinates \( x(t) \), \( y(t) \) and \( z(t) \) are scalar functions of time \( t \). Because the unit vectors are time invariant in the cartesian system, we immediately get the velocity vector (first derivative) and the acceleration vector (second derivative) of the position vector with respect to time:

\[
\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{x}(t)\, e_x + \dot{y}(t)\, e_y + \dot{z}(t)\, e_z \quad \text{(B.2)}
\]

\[
\mathbf{a}(t) = \ddot{\mathbf{r}}(t) = \ddot{x}(t)\, e_x + \ddot{y}(t)\, e_y + \ddot{z}(t)\, e_z \quad \text{(B.3)}
\]

with components \( v_x = \dot{x} \), \( v_y = \dot{y} \) and \( v_z = \dot{z} \), also \( a_x = \ddot{x} \), \( a_y = \ddot{y} \) and \( a_z = \ddot{z} \).

The velocity vector is always tangential to the path of the particle \( P \). In general, this is not true for the acceleration vector.

Figure B.1: Cartesian Coordinate System
The magnitude of the position, velocity and acceleration vector is:

\[ r = | \mathbf{r} | = \sqrt{x^2(t) + y^2(t) + z^2(t)} \]  
(B.4)

\[ v = | \mathbf{v} | = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} \]  
(B.5)

\[ a = | \mathbf{a} | = \sqrt{\ddot{x}^2(t) + \ddot{y}^2(t) + \ddot{z}^2(t)} \]  
(B.6)

### B.2 THREE-DIMENSIONAL MOTION IN CYLINDRICAL COORDINATES

The description of motion on a curved path might be easier described in cylindrical coordinates than in cartesian coordinates.

The position vector is in cylindrical coordinates

\[ \mathbf{r}(t) = r(t) \mathbf{e}_r + z(t) \mathbf{e}_z \]  
(B.7)

whereas the unit vector \( \mathbf{e}_r \) depends on the angle \( \varphi \) and this on his part again depends on the time, which can be written as \( \mathbf{e}_r = \mathbf{e}_r(\varphi(t)) \). Due to this time dependence of the unit vector the derivations with respect to time for the velocity and the acceleration are not as simple expressions as in cartesian coordinates. In fact the following expression can be obtained (without derivation):

\[ \mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{r}(t) \mathbf{e}_r + r(t) \dot{\varphi}(t) \mathbf{e}_\varphi + \dot{z}(t) \mathbf{e}_z \]  
(B.8)

with the velocity components:

\[ v_r = \dot{r} \quad \text{Radial velocity} \]
\[ v_\varphi = r \dot{\varphi} \quad \text{Circular velocity} \]
\[ v_z = \dot{z} \quad \text{Axial velocity} \]

![Figure B.2: Cylindrical Coordinates](image-url)
and the acceleration is:

\[ \mathbf{a}(t) = \ddot{\mathbf{r}}(t) = \left[ \ddot{r}(t) - r(t) \dot{\phi}(t)^2 \right] \mathbf{e}_r + \left[ 2\dot{r}(t) \dot{\phi}(t) + r(t) \ddot{\phi}(t) \right] \mathbf{e}_\phi + \ddot{z}(t) \mathbf{e}_z \]  

(B.9)

with the components:

\[ a_r = \ddot{r} - r \dot{\phi}^2 \quad \text{Radial acceleration} \quad (B.10) \]
\[ a_\phi = 2\dot{r} \dot{\phi} + r \ddot{\phi} \quad \text{Circular acceleration} \quad (B.11) \]
\[ a_z = \ddot{z} \quad \text{Axial acceleration} \quad (B.12) \]

Whereas the component \(-r \dot{\phi}^2 = -r \omega^2\) is the centripetal acceleration and \(\omega\) is the angular velocity. The component \(2\dot{r} \dot{\phi} = 2v_\phi \omega\) is the Coriolis acceleration. The magnitude of velocity and acceleration can be obtained using the rule of vector operation:

\[ |v| = \sqrt{v_r^2 + v_\phi^2 + v_z^2} \quad (B.13) \]
\[ |a| = \sqrt{a_r^2 + a_\phi^2 + a_z^2} \quad (B.14) \]

The magnitude of the position vector may not be mistaken with the radial coordinate \(r\)

\[ |r| = \sqrt{r^2 + z^2} \quad (B.15) \]

### B.3 Natural Coordinates, Intrinsic Coordinates or Path Variables

If the path and the velocity with which the particle \(P\) moves on the path are given, it is of benefit to use path coordinates (so called “natural” coordinates). The coordinate along the path is denoted by \(s = s(t)\) and the path velocity by

![Figure B.3: Visualisation of the single velocity and acceleration components](image_url)
\[ v(t) = \dot{s}(t) \]. The accompanying coordinate system is moving with the particle \( P \) on the path. The time variant tangent unit vector \( \mathbf{e}_t \) always points in the direction of the current motion and is always tangential to the curve. Since the velocity vector is always tangent to the path, the velocity vector can only be expressed in terms of the tangent vector:

\[ v(t) = v(t) \mathbf{e}_t = \dot{s} \mathbf{e}_t \tag{B.16} \]

Approximating the path through a circle in the point \( P \) (the radius of the circle \( \rho \) is variable, in general), the normal unit vector points to the center of this circle. The binormal \( \mathbf{e}_b \) is perpendicular to the osculating plane. It can be obtained using the cross product of the tangent unit vector and the normal unit vector:

\[ \mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n \tag{B.17} \]

The acceleration can be obtained by differentiation of \( v(t) \) with respect to time. Note that the tangent unit vector is time variant and has to be taken in consideration for the derivative of \( v(t) \). With the aid of the Frenet’s Formula of the circle theory, the acceleration is:

\[ a(t) = \ddot{v} \mathbf{e}_t + \frac{v^2}{\rho} \mathbf{e}_n \tag{B.18} \]

with the components

\[ a_t = \dot{v} = \ddot{s} \quad \text{tangential acceleration, path acceleration} \tag{B.19} \]

\[ a_n = \frac{v^2}{\rho} = \frac{s^2}{\rho} \quad \text{normal acceleration} \tag{B.20} \]
You may recognize that for a linear motion with a circle of infinite large radius
the equations derived earlier can be obtained.
C.1 Special Cases for the Calculation of the Angular Momentum

In general, the angular momentum in a *-reference frame is given by eq. (3.91)

\[ L_{0*} = L_0 - r_{0*} \times p - m r_S^* \times \bar{v}_{0*} \]

We distinguish the following special cases

1. 0* is a fixed point:
then the velocity of this point is \( \bar{v}_{0*} = 0 \) which results:

\[ L_{0*} = L_0 - r_{0*} \times p \]

2. 0* coincides with the point \( A \) which belongs to the moving rigid body, and moves with it:

\[ L_{0*} = L_A = L_0 - r_A \times p - m r_A S \times \bar{v}_A \]

mit \( r_{0*} = r_A \) und \( p = m \bar{v}_S \).

3. 0* coincides with the center of gravity of the rigid body \( A = S \):
then \( r_{AS} = 0 \) and \( \bar{v}_A = \bar{v}_S \) which means:

\[ L_{0*} = L_S = L_0 - r_S \times p \]

As the angular momentum \( L_0 \) has two terms, translation and rotation, such as

\[ L_0 = r_S \times m \bar{v}_S + J_S \bar{\omega} \]

this will reduce the angular momentum at the center of gravity of the rigid body \( S \) to just the rotational term

\[ L_S = J_S \bar{\omega} \].
4. $0^*$ is the instantaneous center of rotation $A$:

at the instantaneous center of rotation, the velocity is given as $v_A = 0$
and this point is not fixed in space, neither fixed to the body, but its
motion is described by the path of the instantaneous center of rotation.

Using the translational and rotational terms of the angular momentum
as in 3, and using $v_A = 0$, we get:

$$L_A = I_A \omega$$

This final result is also valid if the point $A$ is fixed point.

Effects on the Angular momentum theorem

It is now important to keep in mind what are the terms that change in the
angular momentum theorem with the change of reference point.

Starting with the angular momentum theorem with reference point 0 (see eq. (3.84)):

$$L_0 = M_0$$

With the change of reference point to an arbitrary moving point $0^*$ the angular
momentum is written according to eq. (3.105) as

$$L_{0^*} = L_0 - r_{0^*} \times p - m r_{0^*}^s \times v_{0^*}$$

and the moment with regard to the new reference point is:

$$M_{0^*} = r^* \times F = \left(r - r_{0^*}\right) \times F$$
$$M_{0^*} = M_0 - r_{0^*} \times F$$

which leads to the following angular momentum theorem applied at reference
point 0:

$$\frac{d}{dt} \left[ L_{0^*} + r_{0^*} \times p + m r_{0^*}^s \times v_{0^*} \right] = M_{0^*} + r_{0^*} \times F$$

(C.1)

Calculating the time derivative and noticing that $\dot{p} = F$ and

$$r_{0^*} \times \dot{p} = r_{0^*} \times F,$$

which will result

$$\dot{L}_{0^*} + \dot{r}_{0^*} \times p + m \dot{r}_{0^*}^s \times v_{0^*} + m \dot{r}_{0^*}^s \times \dot{v}_{0^*} = M_{0^*}.$$  (C.2)

Moreover, we have

$$v_S = v_{0^*} + v_{0^*}^s$$

and

$$\dot{r}_{0^*}^s = \dot{v}_{0^*}^s = v_S - v_{0^*}.$$
so the third term on the left side becomes:
\[
m \dot{r}^* \times v^* = m (v_S - v^*) \times v^* = m v_S \times v^* - m v^* \times v^* = m v_S \times v^* = -v^* \times m v_S = -v^* \times \mathbf{p}
\]

Finally, we have:
\[
\dot{L}^*_0 + m r^*_S \times \dot{v}^*_0 = M_0^*
\]

It is clear from this last relation, that a new term is introduced to the angular momentum theorem when it is written in term of the moving point '0*', and the new term, the second term in the left side in eq. (C.3), should always be get in mind. At this point, different special cases could be observered.

1. The reference point 0* is moving with constant velocity:
   as \( v^*_0 = \text{konst} \) then \( \dot{v}^*_0 = 0 \) and
   \[\dot{L}^*_0 = M_0^*\]

2. The reference point 0* is the center of gravity: \( 0^* = S \):
   \( S \) could also perform accelerated motion. As \( r^*_S = 0 \), then
   \[\dot{L}^*_S = M_S\]

3. The point 0* is fixed: \( 0^* = A \) or it is the instantaneous center of rotation:
   for a fixed point or the instantaneous center of rotation we have: \( v_A = 0 \)
   which results
   \[\dot{L}^*_A = M_A\]

Normally the center of gravity or the instantaneous center of rotation is used in solving problems of classical mechanics, which seems to be a good choice as the second term in the left side in eq. (C.3) will disappear.

As we have seen, the angular momentum has two terms, the translational and the rotational. Choosing the center of gravity, or a fixed point (i.e. instantaneous center of rotation) as reference point, this will reduce the angular momentum to the rotational part, and the angular momentum theorem based on such choice takes the following form
\[
\dot{L}^*_S = \frac{d}{dt} (L^*_S \omega) = M_S
\]
similarly,

\[ \dot{L}_A = \frac{d}{dt} (J_A \omega) = M_A \quad (C.5) \]

**Note** The term \( \frac{d}{dt} (J_S \omega) \) does not simply equal to \( J_S \dot{\omega} \), as the moment of inertia of a rotating rigid body in the a fixed reference frame can be time dependent. An exception to this is, for example, a rotating cylinder about its axis that is coincides with one of the axis of the reference frame, in this situation, we could write that

\[ \frac{d}{dt} (J_S \omega) = J_S \dot{\omega} \]

The dependency of the moment of inertia on the actual position (angle) is a deciding factor to the easy and direct application of the angular momentum theorem. Because of this body-fixed reference frame is used, as the moment of inertia in the body-fixed frame stays constant and the time derivative of the moment of inertia is canceled!
D.1 EXCITATION WITH CONSTANT AMPLITUDE OF FORCE - COMPLEX APPROACH

Let us first recall that we can represent a real harmonic functions by a complex exponential function using

\[ e^{i\Omega t} = \cos \Omega t + i \sin \Omega t \]  
(D.1)

From this we can derive that

\[ \cos \Omega t = \frac{e^{i\Omega t} + e^{-i\Omega t}}{2} \]  
(D.2)

and

\[ \sin \Omega t = \frac{e^{i\Omega t} - e^{-i\Omega t}}{2i} \]  
(D.3)

The harmonic force is:

\[ F(t) = \hat{F} \cos \Omega t = \frac{\hat{F}}{2} \left( e^{i\Omega t} + e^{-i\Omega t} \right) = \frac{\hat{F}}{2} e^{i\Omega t} + \frac{\hat{F}}{2} e^{-i\Omega t} \]  
(D.4)

This means that we have to solve the equation of motion twice, for the \( e^{i\Omega t} \) and the \( e^{-i\Omega t} \) term. For the first step we make the approach:

\[ x_1(t) = \hat{x}_1 e^{i\Omega t} \]  
(D.5)

\[ x_2(t) = \hat{x}_2 e^{-i\Omega t} \]  
(D.6)

Putting both approaches into the equation of motion yields:

\[ (-\Omega^2 m + i\Omega c + k) \hat{x}_1 e^{i\Omega t} = \frac{\hat{F}}{2} e^{i\Omega t} \]  
(D.7)

\[ (-\Omega^2 m - i\Omega c + k) \hat{x}_2 e^{i\Omega t} = \frac{\hat{F}}{2} e^{-i\Omega t} \]  
(D.8)

Dividing by \( k \) and introducing the frequency ratio \( \eta \) (eq. (4.102))

\[ \left[ (1 - \eta^2) + 2D\eta \right] \hat{x}_1 = \frac{\hat{F}}{2k} \]  
(D.9)

\[ \left[ (1 - \eta^2) - 2D\eta \right] \hat{x}_2 = \frac{\hat{F}}{2k} \]  
(D.10)
The solution for \( x_1 \) and \( x_2 \) are:

\[
\hat{x}_1 = \frac{(1 - \eta^2) - 2D\eta \hat{F}}{(1 - \eta^2)^2 + 4D^2\eta^2 \frac{2}{k}} \tag{D.11}
\]

\[
\hat{x}_2 = \frac{(1 - \eta^2) + 2D\eta \hat{F}}{(1 - \eta^2)^2 + 4D^2\eta^2 \frac{2}{k}} \tag{D.12}
\]

As we can see, the solution of one part is the conjugate complex of the other:

\[
\hat{x}_1 = \hat{x}_2^* \tag{D.13}
\]

The solution for \( x(t) \) is combined from the two partial solutions, which we just have found:

\[
x(t) = \hat{x}_1 e^{i\Omega t} + \hat{x}_2 e^{-i\Omega t} \tag{D.14}
\]

This can be resolved:

\[
x(t) = \hat{x}_1 \cos \Omega t + i\hat{x}_1 \sin \Omega t + \hat{x}_2 \cos \Omega t + i\hat{x}_2 \sin \Omega t \tag{D.15}
\]

and using eq. (D.13), we finally get:

\[
x(t) = 2 \text{Re} \{\hat{x}_1\} \cos \Omega t - 2 \text{Im} \{\hat{x}_1\} \sin \Omega t \tag{D.16}
\]

The factor of 2 compensates the factor \( \frac{1}{2} \) associated with the force amplitude. All the information can be extracted from \( \hat{x}_1 \) only so that only this part of the solution has to be solved:

\[
x(t) = \frac{(1 - \eta^2)}{(1 - \eta^2)^2 + 4D^2\eta^2} \frac{\hat{F}}{k} \cos \Omega t + \frac{2D\eta}{(1 - \eta^2)^2 + 4D^2\eta^2} \frac{\hat{F}}{k} \sin \Omega t \tag{D.17}
\]

which is the same result as eq. (4.91) with eq. (4.103) and eq. (4.104).

Also the magnitude \( \hat{x} = |\hat{x}_1| \) and phase \( \varphi \) can be obtained in the same way and yield the previous results:

Magnitude:

\[
\hat{x} = \frac{1}{\sqrt{(1 - \eta^2)^2 + 4D^2\eta^2}} \frac{1}{k} \hat{F} = V_1(\eta, D) \frac{1}{k} \hat{F} \tag{D.18}
\]

Phases:

\[
\tan \varphi = -\frac{\text{Im} \{\hat{x}_1\}}{\text{Re} \{\hat{x}_1\}} = \frac{2D\eta}{1 - \eta^2} \tag{D.19}
\]
D.2 EXCITATION WITH CONSTANT AMPLITUDE OF FORCE - ALTERNATIVE COMPLEX APPROACH

Instead of eq. (D.4), we can write

\[ F(t) = \hat{F} \cos \Omega t = \frac{\hat{F}}{2} (e^{i\Omega t} + e^{-i\Omega t}) = 2 \text{Re} \left\{ \frac{\hat{F}}{2} e^{i\Omega t} \right\} \tag{D.20} \]

or

\[ F(t) = \hat{F} \cos \Omega t = \text{Re} \left\{ \hat{F} e^{i\Omega t} \right\} \tag{D.21} \]

According to this approach, we formulate the steady state response as

\[ x(t) = \text{Re} \left\{ \hat{X} e^{i\Omega t} \right\} \tag{D.22} \]

The complex amplitude \( \hat{X} \) is determined from the equation of motion, solving

\[ \text{Re} \left\{ ( -\Omega^2 m + i\Omega c + k ) \hat{X} e^{i\Omega t} \right\} = \text{Re} \left\{ \hat{F} e^{i\Omega t} \right\} \tag{D.23} \]

The real parts are equal if the complex expression is equal:

\[ ( -\Omega^2 m + i\Omega c + k ) \hat{X} e^{i\Omega t} = \hat{F} e^{i\Omega t} \tag{D.24} \]

Elimination of the time function yields:

\[ ( -\Omega^2 m + i\Omega c + k ) \hat{X} = \hat{F} \tag{D.25} \]

The expression in brackets is also called the dynamic stiffness

\[ k_{\text{dyn}}(\Omega) = (k - \Omega^2 m + i\Omega) \tag{D.26} \]

Now we solve eq. (D.25) to get the complex amplitude:

\[ \hat{X} = \frac{\hat{F}}{(-\Omega^2 m + i\Omega c + k)} \tag{D.27} \]

The expression

\[ H(\Omega) = \frac{1}{(-\Omega^2 m + i\Omega c + k)} = \frac{\hat{X}}{\hat{F}} = \frac{\text{Output}}{\text{Input}} \tag{D.28} \]

is the complex Frequency Response Function (FRF). Introducing the dimensionless frequency \( \eta \) as before yields:

\[ \hat{X} = \frac{1}{(1 - \eta^2 + i2D\eta)} \frac{\hat{F}}{k} \tag{D.29} \]
Because
\[ \hat{x} \cos(\Omega t - \varphi) = \hat{x} \Re \{ e^{i(\Omega t - \varphi)} \} = \Re \{ \hat{x} e^{-i\varphi} e^{i\Omega t} \} = \Re \{ \hat{X} e^{i\Omega t} \} \] (D.30)
we take the magnitude \( \hat{x} \) and phase lag \( \varphi \) of this complex result
\[ \hat{X}(\Omega) = \hat{x} e^{-i\varphi} \] (D.31)
which leads to the same result as before, see eq. (D.18) and eq. (D.19):
\[ \hat{x} = \frac{1}{\sqrt{(1 - \eta^2)^2 + 4D^2\eta^2}} \frac{1}{k} \hat{F} = V_1(\eta, D) \frac{1}{k} \hat{F} \] (D.32)
\[ \tan \varphi = -\frac{\Im \{ \hat{x}_1 \}}{\Re \{ \hat{x}_1 \}} = \frac{2D\eta}{1 - \eta^2} \] (D.33)

D.3 FOURIER SERIES - ALTERNATIVE REAL REPRESENTATION

We can write the Fourier series as a sum of cosine functions with amplitude \( c_k \) and a phase shift \( \varphi_k \):
\[ x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega t + \varphi_k) \] (D.34)
\[ c_k = \sqrt{a_k^2 + b_k^2} \quad \text{and} \quad \varphi_k = \arctan \left( -\frac{b_k}{a_k} \right) \] (D.35)

D.4 FOURIER SERIES - ALTERNATIVE COMPLEX REPRESENTATION

The real trigonometric functions can also be transformed into complex exponential expression:
\[ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jkw t} \] (D.36)
The \( X_k \) are the complex Fourier coefficients which can be determined by solving the integral:
\[ X_k = \frac{1}{T} \int_0^T x(t) e^{-jkw t} dt \] (D.37)
or

\[ X_k = \frac{1}{T} \int_0^T x(t) \left[ \cos k\omega t - i \sin k\omega t \right] dt \]  \hspace{1cm} (D.38)

which clearly shows the relation to the real Fourier coefficients series given by eq. (4.142) and eq. (4.143):

\[ \text{Re} \{X_k\} = \frac{a_k}{2}; \quad \text{Im} \{X_k\} = -\frac{b_k}{2} \]  \hspace{1cm} (D.39)

The connection to the other real representation (chap. 4.7.1) is:

\[ |X_k| = c_k \quad \tan \varphi_k = \left( \frac{\text{Im} \{X_k\}}{\text{Re} \{X_k\}} \right) \]  \hspace{1cm} (D.40)

The coefficients with negative index are the conjugate complex values of the corresponding positive ones:

\[ X_{-k} = X_k^* \]  \hspace{1cm} (D.41)
D.5 Magnification Functions

![Figure D.1: Magnification function $V_1$](image1)

$$V_1 = \frac{1}{\sqrt{(1-\eta^2) + 4D\eta^2}}$$

![Figure D.2: Magnification function $V_2$](image2)

$$V_2 = \frac{\sqrt{1+4D\eta^2}}{\sqrt{(1-\eta^2) + 4D\eta^2}}$$
D.5 magnification functions

$$V_3 = \frac{\eta^2}{\sqrt{(1-\eta^2)^2 + 4D^2\eta^2}}$$

Figure D.3: Magnification function $V_3$

$$V_4 = \frac{\eta^2 \sqrt{1+4D^2\eta^2}}{\sqrt{(1-\eta^2)^2 + 4D^2\eta^2}}$$

Figure D.4: Magnification function $V_4$
Figure D.5: Phase $\varphi$ - for magnification function $V_1$ and $V_3$

Figure D.6: Phase $\varphi$ - for magnification function $V_2$ and $V_4$
Die hier vorliegende Zusammenstellung maschinendynamischer Themen umfasst grundlegende Prinzipien und Arbeitsgebiete. Die dargestellten Inhalte lassen sich in vielen Lehrbüchern wiederfinden. Im besonderen sei jedoch auf die folgenden Werke hingewiesen:

FORMULARY
\[
\begin{align*}
\sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\
\cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \\
\sin(2\varphi) &= 2 \sin \varphi \cos \varphi \\
\cos(2\varphi) &= \cos^2(\varphi) - \sin^2(\varphi) = 1 - 2\sin^2(\varphi)
\end{align*}
\]

Table 2: Trigonometric functions

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Equivalent stiffness (bending stiffness (EI = \text{const.}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ F = k_{eq}w_{\text{max}} ]</td>
<td>(k_{\text{ers}} = 48\frac{EI}{l^5})</td>
</tr>
<tr>
<td>[ F = k_{eq}w_{\text{max}} ]</td>
<td>(k_{\text{ers}} = 12\frac{EI}{l^5})</td>
</tr>
<tr>
<td>[ F = k_{eq}w_{\text{max}} ]</td>
<td>(k_{\text{ers}} = 3\frac{EI}{l^5})</td>
</tr>
</tbody>
</table>

Table 3: Equivalent stiffnesses for different loaded beams
\((1 \pm x)^m = 1 \pm mx + \frac{m(m-1)}{2!}x^2 \pm \frac{m(m-1)(m-2)}{3!}x^3 + \ldots \quad m > 0\)

\[+ (\pm1)^n \frac{m(m-1) \ldots (m-n+1)}{n!}x^n + \ldots\]

\[\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{3}x^2 - \ldots \quad m = \frac{1}{2}\]

Table 4: Reihenentwicklung: Binomische Reihe mit positiven Exponenten \((|x| \leq 1)\)

<table>
<thead>
<tr>
<th>Springs connection</th>
<th>Equivalent stiffness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallelschaltung</td>
<td></td>
</tr>
<tr>
<td>(k_i)</td>
<td>(k_{eq} = \sum_{i=1}^{n} k_i)</td>
</tr>
<tr>
<td></td>
<td>(n = 2:)</td>
</tr>
<tr>
<td></td>
<td>(k_{eq} = k_1 + k_2)</td>
</tr>
</tbody>
</table>

| Reihenschaltung     |                     |
| \(k_i\)            | \(\frac{1}{k_{eq}} = \sum_{i=1}^{n} \frac{1}{k_i}\) |
|                     | \(n = 2:\)          |
|                     | \(\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}\) |
| \(k_2\)            | \(k_{eq} = \frac{k_1k_2}{k_1+k_2}\) |
| \(F\)              |                     |

Table 5: Equivalent stiffness of springs in parallel and serie connection
Rigid body | Mass moment of inertia
---|---
Quader | 

\[
J_{xx} = \frac{1}{12} m (b^2 + h^2) \\
J_{yy} = \frac{1}{12} m (h^2 + l^2) \\
J_{zz} = \frac{1}{12} m (l^2 + b^2)
\]

Stab \((b, h << l)\) | 

\[
J_{yy} = J_{zz} = \frac{1}{12} ml^2
\]

Zylinder | 

\[
J_{xx} = J_{yy} = \frac{1}{12} m (l^2 + 3r^2) \\
J_{zz} = \frac{1}{2} mr^2
\]

Scheibe \((h << r)\) | 

\[
J_{xx} = J_{yy} = \frac{1}{4} mr^2 \\
J_{zz} = \frac{1}{2} mr^2
\]

Hohlzylinder | 

\[
J_{xx} = J_{yy} = \frac{1}{4} m \left( R^2 + r^2 + \frac{h^2}{3} \right) \\
J_{zz} = \frac{1}{2} m \left( R^2 + r^2 \right)
\]

Kreisringscheibe \(R \approx r, h << R\) | 

\[
J_{xx} = J_{yy} = \frac{1}{2} m R^2 \\
J_{zz} = m R^2
\]

Kugel | 

\[
J_{xx} = J_{yy} = J_{zz} = \frac{2}{5} mr^2
\]

Table 6: Mass moment of inertia (given at the center of gravity \(S\))