Cyclic Stress-Strain Response of Metals and Alloys Modelling on a Microstructural Basis

Summary: The observation of a heterogeneous dislocation distribution established in metals and alloys under cyclic-loading conditions indicates a locally varying yield stress. Hence, it seems to be justified to consider even a single-phase material as a composite consisting of elements with different flow stresses. This idea is used in order to describe and predict the cyclic stress-strain behaviour of metallic materials. The range of applicability of multi-component models with respect to the dislocation slip character and the loading conditions is defined. A statistical mathematical treatment leads to equations which allow one to determine the distribution function of yield stresses directly from a single branch of a hysteresis loop. The procedure of the stress-strain path calculation is illustrated by means of examples and a comparison of calculated and experimentally obtained results is presented.

1. INTRODUCTION

If metallic materials are subjected to cyclic loading conditions, a cyclic stress-strain response (CSS-response) results which is in uniaxial laboratory testing mostly recorded in the form of stress-strain hysteresis loops. Each hysteresis loop represents the path of the stress $\sigma$ versus the strain $\varepsilon$ for a single cycle. Since the elastic part of the strain is considered as not contributing to the damage evaluation typical of fatigue, often the plastic strain $\varepsilon_{pl}$ is plotted along the abscissa instead of the total mechanical strain. The plastic strain can simply be calculated from the total strain using Hooke’s law for the elastic part of the strain $\varepsilon_{el}$.

$$\varepsilon_{pl} = \varepsilon - \varepsilon_{el} = \varepsilon - \frac{\sigma}{E}$$  \hspace{1cm} (1)

Important characteristic quantities (such as the Young’s modulus $E$, the minimum and maximum strains $\varepsilon_{\min}$ and $\varepsilon_{\max}$, the minimum and maximum stress
\( \sigma_{\text{min}} \) and \( \sigma_{\text{max}} \), the ranges of strain \( \Delta \varepsilon \), of stress \( \Delta \sigma \) and of plastic strain \( \Delta \varepsilon_{\text{pl}} \) can be taken from the hysteresis loop and are commonly used for continuative representations such as the cyclic deformation curve and the CSS curve. Besides these “primary” quantities, the hysteresis loop contains also the integral information on the microscopic deformation processes taking place during cyclic plastic deformation. It is the intention of this paper to provide a physically sound basis for an interpretation of the hysteresis loop from a microstructural point of view. This is done by means of the introduction, mathematical treatment and application of simple multi-component models [1]. The models used are intentionally kept very simple, in order to illustrate the basic concepts and to avoid the need of fitting parameters.

A first insight into the foundations of the cyclic deformation behaviour as compared to the monotonic loading can be gained by means of the introduction of two well-known phenomena. The first phenomenon is depicted in Fig.1 and is commonly termed the memory of prior load history. If a metallic material is subjected to a stress increase up to a stress level B (see Fig.1), an unloading down to level C and a subsequent reloading to a stress E exceeding the prior maximum, the stress-strain path shows a somewhat surprising feature. The reloading leads to the formation of a small hysteresis loop which is closed at level D (equal to B) and the material seem to recall the prior monotonic deformation at this point. The stress-strain path kinks at D and forms a continuation of the monotonic stress-strain curve A-B.

The second phenomenon is called Masing’s hypothesis or sometimes also 1:2-rule and illustrated in Fig.2. If a material is in a microstructurally stable
condition (e.g., by means of cyclic deformation until a steady-state condition is reached), a simple correlation exists between the monotonic stress-strain curve (or the initial half cycle of cyclic loading) and the ascending branch of the stabilized hysteresis loop. If the hysteresis loop branch is plotted in so-called relative co-ordinates $\epsilon_r$ and $\sigma_r$, i.e., the last load reversal point is defined as the origin of the stress-strain co-ordinate system, and the monotonic curve is magnified by a factor of two, both curves coincide in a reasonable approximation. In other words, both stress-strain paths are geometrically similar with a scale factor of two.

![Fig.2. Masing's hypothesis or 1:2-rule](image)

The phenomena described are well-known and implemented in many concepts used for cycle counting and fatigue life prediction. Within the scope of this paper it is more important to note that these characteristic features of cyclic deformation indicate the composite character of monolithic metallic materials. The use of multi-component models for the prediction of the cyclic stress-strain response reproduces these phenomena implicitly, as will be shown later.

2. SIMPLE MULTI-COMPONENT MODELS

2.1. Microstructural Foundation
Cyclic deformation often leads to the establishment of a steady-state dislocation arrangement, which manifests itself in a state of cyclic saturation. The dislocations distribution of this steady-state condition is characterized by a very
pronounced local heterogeneity. Figure 3 shows as an example the steady-state
dislocation arrangement formed in copper as a consequence of cyclic plastic
deformation at a plastic strain amplitude $\Delta \epsilon_{pl}/2$ of $2 \times 10^{-4}$. At this low amplitude,
single slip prevails and persistent slip bands are formed which can clearly be
seen in Fig.3 because of their ladder-like appearance. The persistent slip bands
are embedded in a matrix structure which consists of dislocation-rich bundles or
veins and dislocation-poor channels. Irrespective of the details of the dislocation
arrangement established, the dislocation density varies strongly with the
location. Consequently, locally different flow stresses must exist, since the
dislocation density determines the flow stress.

Fig.3. Dislocation arrangement in polycrystalline copper in cyclic saturation established
at a plastic strain amplitude of 0.02% [2]. The arrows mark secondary dislocations.

2.2. Mathematical Treatment
Probably the simplest way, how to treat this local variation in yield stress, was
already proposed in 1923 by Masing [3]. The idea is based on the assumption
that a material must be considered as a composite consisting of elements which
deform in an ideal elastic-plastic way and which are arranged in parallel. Figure
4 give a schematic representation of this concept.
In a statistical treatment of Masing’s model according to Afanasev [4], Holste
and Burmeister [5] and Polák and Klesnil [6], the distribution of the microscopic flow stresses $\sigma_{if}$ of the elements $i$ is described by a probability density function $f_p(\sigma_{if})$ which fulfils the equation

$$\int_{0}^{\infty} f_p(\sigma_{if}) d\sigma_{if} = 1 \quad (2)$$

Fig. 4. Schematic representation of the parallel arrangement of ideal elastic-plastic elements according to the model of Masing

The probability (or area fraction) of elements with critical flow stresses in an interval between $\sigma_{if}^*$ and $\sigma_{if}^*$ is $f_p(\sigma_{if}^*) d\sigma_{if}$. The total stress acting on a metal can be calculated from the local stresses $\sigma_i$ in the individual elements with the equation

$$\sigma = \int_{0}^{\infty} f_p(\sigma_{if}) \sigma_{if} d\sigma_{if} \quad (3)$$

Since the stress-strain behaviour of each element is considered to be ideal elastic-plastic and the Young’s modulus $E$ is assumed to be identical for all elements, the microscopic stress $\sigma_i$ can be expressed in terms of the total strain $\varepsilon$ and the element-specific yield strength $\sigma_{if}$.

$$\sigma_i = \begin{cases} E\varepsilon & \text{for } |E\varepsilon| \leq \sigma_{if} \\ \sigma_{if} & \text{for } E\varepsilon > \sigma_{if} \\ -\sigma_{if} & \text{for } E\varepsilon < -\sigma_{if} \end{cases} \quad (4)$$
From Eqs.(3) and (4) an expression for the monotonic stress-strain curve can be derived by dividing the integral in Eq.(2) into two parts. The first term represents the elements in which the local yield stress is reached, the second term corresponds to those elements which are deformed purely elastically.

\[
\sigma = \int_{0}^{E} f_p(\sigma_{\text{y}}) \sigma_{\text{y}} d\sigma_{\text{y}} + E \epsilon \int_{\text{e}}^{g} f_p(\sigma_{\text{y}}) d\sigma_{\text{y}}
\]

(5)

By means of the second integral function of the probability density function

\[
G_p(x) = \int_{0}^{x} f_p(x') dx'dx'
\]

(6)

Eq.(5) can be brought into the form

\[
\sigma = E \epsilon - G_p(E \epsilon)
\]

(7)

For the description of the cyclic stress-strain behaviour, it is necessary to use relative co-ordinates as introduced above. A derivation similar to the monotonic case described above finally leads to the equation

\[
\sigma_r = 2 \left( E \frac{E}{2} - G_p \left( \frac{E \epsilon}{2} \right) \right)
\]

(8)

The comparison of Eq.(7) with Eq.(8) documents that according to Masing’s model the branch of the hysteresis loop is geometrically similar to the monotonic stress-strain curve by a scale factor of two. This is the second phenomenon illustrated in the introduction of this paper.

The most striking advantage of this type of model is that it does not contain any fit parameter. The calculation of the cyclic stress-strain response is solely carried out on the basis of the distribution function of the microscopic yield stress \(f_p(\sigma_{\text{y}})\). A double differentiation of Eq.(8) easily shows that this function, which may be considered a mathematical representation of the microstructure, can be obtained directly from a single branch of a stress-strain hysteresis loop in relative co-ordinates according to:

\[
f_p(\sigma_{\text{y}}) = -2 \frac{d^2 \sigma_r}{E \frac{d \epsilon_r}{2}}
\]

(9)

One important consequence resulting from Masing’s model is that under the assumption of a constant \(f_p(\sigma_{\text{y}})\) the ascending (descending) branches of hysteresis loops of different amplitudes translated in such a way that their tips coincide at the position of minimum (maximum) stress follow a common curve. Today, this feature is called Masing behaviour and used for a graphic verification disregarding other consequences of Masing’s model.

It should be emphasised at this point that the strict parallel arrangement is not a prerequisite for the successful explanation of many important cyclic deformation behavioural patterns, such as the 1:2-rule, the Bauschinger effect
and the memory of prior load history. It is shown in [1] in detail that a plain serial arrangement of elastic linear-plastic elements leads to basically identical results.

3. APPLICABILITY TO CONSTANT-AMPLITUDE CYCLIC LOADING

3.1. Slip Character and Dislocation Arrangement
The slip character of a material is a basic parameter that determines the type of dislocation arrangement formed during cyclic loading and therefore also the cyclic stress-strain response. The term slip character describes the tendency to form a three-dimensional dislocation arrangement. The two extreme cases of slip character are on the one hand a pure planar dislocation slip and on the other hand a pure wavy dislocation movement.

![Fig.5. Dislocation arrangement as a function of loading amplitude and slip character for fcc single-crystalline metals](image)

The results of various studies on face-centred cubic (fcc) metals and alloys are summarized in Fig.5. The different types of dislocation arrangements are mapped as a function of the number of cycles to failure \( N_f \) (abscissa) and the slip character of the material (ordinate). \( N_f \) depends on the loading amplitude, in the sense that a high plastic strain range \( \Delta \varepsilon_{pl} \) corresponds to a low cycle number to failure, and vice versa. It is assumed that a state of cyclic saturation is established at each loading amplitude. The representation shown as Fig.5, which was first introduced in 1968 by Feltner and Laird [7] and later on confirmed quantitatively by Lukáš und Klesnil [8], roughly classifies materials into those exhibiting wavy slip and
those showing planar slip. In planar-slip metals and alloys, the dislocation motion is confined to the slip plane. Planar arrangements form, mainly consisting of edge dislocations that occupy a slip plane and line up parallel to each other. Secondary slip contributes to cyclic deformation at high amplitudes. The dislocation arrangement in cyclic saturation of wavy-slip metals depend very strongly on the loading amplitude. At low amplitudes (i.e., high value of $N_f$) arrangements of edge dislocation dipoles are found that form mainly due to single slip. Dislocations agglomerate to bundles and veins, which are separated from each other by regions of low dislocation density (channels). Embedded in the matrix, persistent slip bands can form. At high amplitudes, multiple slip takes place and gives rise to labyrinth and cell structures.

3.2. Wavy-Slip Type Materials
As illustrated in Fig.5, cyclic loading at a constant amplitude (single-step test) leads to the formation of a characteristic dislocation arrangement that is in the case of wavy slip very strongly dependent on the amplitude used. Since the probability function can be considered as a mathematical representation of the microstructure, it must be expected that under these conditions no Masing behaviour can be found.

Figure 6 shows the hysteresis loops of polycrystalline copper in cyclic saturation at five different plastic strain amplitudes in relative co-ordinates. Copper can be considered as a typical representative of wavy-slip type metals. Clearly the ascending branches of the loops do not form a common curve, but are individually different. In terms of the model of Masing this observation
means that the probability density function (which is represented in the curvature of the ascending branches) depend strongly on the microstructure established. Hence, this representation does not directly confirm the idea of Masing, but does also not disagree.

3.3. Planar-Slip Type Materials
The change in the dislocation arrangement of planar-slip materials with loading amplitude is much less significant as compared to wavy-slip materials (see Fig.5). Consequently a smaller dependence of the probability density function on the plastic strain amplitude is expected. Figure 7 depicts the hysteresis loops of \(\alpha\)-brass, which shows a very pronounced planarity of dislocation movement, in the Masing representation (i.e., in relative coordinates).

![Hysteresis loops of cyclic saturation of \(\alpha\)-brass in relative co-ordinates](Fig.7)

It is typical of planar-slip metals and alloys that the ascending branches of the loops are rather close together. However, they do not really coincide and hence they cannot be described by a common curve. Therefore a simple application of the multi-component models on the basis of an amplitude-independent \(f_p(\sigma_p)\) seem not to be appropriate (despite that fact that this is sometime done and justified as an engineering approach).

4. APPLICABILITY TO VARYING-AMPLITUDE CYCLIC LOADING
4.1. The Incremental Step Test
The examples reported so far were taken from an extensive study of the cyclic stress-strain behaviour of various materials (more details are given in [1,9,10,11]) and document that the distribution function of the microscopic
yield stresses depends on the loading amplitude. Therefore, Masing behaviour, as defined above, seems to be a rare exception under single-step loading conditions.

From an engineering point of view, the behaviour under variable amplitude loading is more significant, because most mechanically loaded components are subjected to a cyclic loading in service, which is typically a complex random loading. In order to study the behaviour under in-service conditions, as a first approach so-called Incremental Step Tests (IST) were carried out on numerous materials. Figure 8 shows the IST used in this study. The strain block consists of 30 cycles within which the plastic strain amplitude $\Delta \varepsilon_{pl}/2$ is firstly increased and subsequently decreased both linearly in time between limiting amplitudes of $2 \times 10^{-4}$ and $5 \times 10^{-3}$.

The IST was originally introduced as a time- and material-saving method to determine the cyclic stress-strain curve [12]. The idea was that after a few IST loading blocks a stabilized stress-strain path (see Fig.8) results and the CSS curve can then be obtained simply by connecting the load reversal points in
tension. It is important to mention that the CSS curve determined by means of this procedure can strongly deviate from the “classically” determined CSS curve, i.e., the one determined from various single-step tests. This is due to the fact that at a constant amplitude an amplitude-specific microstructure is established, as discussed above. Hence the classic CSS curve represents different microstructures, while the CSS curve obtained in the IST corresponds to the **one** microstructure which is established as a consequence of the continuously varying amplitude.

### 4.2. Wavy-Slip Type Materials

![Dislocation arrangements in polycrystalline copper formed in tests at constant amplitudes (a and b) and in the Incremental Step Test (c)](image)

Fig. 9. Dislocation arrangements in polycrystalline copper formed in tests at constant amplitudes (a and b) and in the Incremental Step Test (c)

Figure 9 compares the dislocation arrangements formed in copper (as a representative of a material exhibiting wavy slip) in the stabilized condition
established in single-step tests at a low and a high $\Delta \varepsilon_p/2$ value and in the IST, which was carried out in such a way that the extreme amplitudes in the loading block are identical to the ones used in single-step tests depicted. Similar to the micrograph shown as Fig.3, a typical single-slip arrangement consisting of a bundle/vein matrix and embedded persistent slip bands can be identified in Fig.9a. As a consequence of high-amplitude loading, a cell structure typical of multiple slip in wavy-slip materials is formed (Fig.9b). The continuous variation of the amplitude in the IST leads to a dislocation arrangement which is independent of the instantaneously acting amplitude (i.e., independent of the amplitude within the strain block, when the test is stopped). The microstructure is very similar to the one formed in constant-amplitude loading applying constantly the maximum amplitude of the IST (compare Fig.9c with Fig.9b). The observation of a amplitude-independent microstructure formed in the IST provides the possibility to check whether Masing’s model holds true. If $f_p(\sigma)$ has a physical meaning, this function should represent the microstructure and should therefore also be unaffected by the amplitude acting in the IST.

In Fig.10 hysteresis loops extracted from the IST stress-strain path are plotted in relative co-ordinates for a low-carbon steel which exhibits wavy slip. Clearly, the ascending branches form a perfect common curve. Since the probability density function can directly be calculated from the ascending stress-strain course (by means of Eq.(9)), $f_p(\sigma)$ is identical for all amplitudes.

### 4.3. Planar-Slip Type Materials

The change in the dislocation arrangement in planar-slip type materials with straining amplitude is much less pronounced as compared to metals and alloys.
of wavy-slip character. Nevertheless, there is a change, which is illustrated in Fig.11 for α-brass. At a high plastic strain amplitude of $\Delta \varepsilon_{pl}/2 = 5 \times 10^{-3}$ clearly a second slip system is activated (Fig.11b), while the dislocations at $\Delta \varepsilon_{pl}/2 = 2 \times 10^{-4}$ are lined up on parallel slip planes of the same type (single-slip, Fig.11a). Again, the IST leads to a stabilized dislocation arrangement (Fig.11c) which is independent of the amplitude which is acting when the test is stopped. This microstructure is again very similar to the microstructure formed, when the maximum amplitude is permanently applied in a single-step test (Fig.11b).

![Dislocation arrangements in α-brass formed in tests at constant amplitudes (a and b) and in the Incremental Step Test (c)](image)

Despite this finding of a constant microstructure established in the IST, the Masing representation of the IST hysteresis loops does not show a common ascending hysteresis branch. In order to make sure that microstructural changes, which might occur within a single strain block of the IST, are definitely excluded, a test was carried out, in which the material was single-amplitude...
loaded at \( \Delta \varepsilon_p / 2 = 5 \cdot 10^{-3} \), and after attaining a stabilized condition the continuous cycling was interrupted by single cycles at lower amplitudes. The resulting hysteresis loops are shown in Fig.12 indicating that even under these strict conditions the ascending branches differ from each other.

![Fig.12. Hysteresis loops of \( \alpha \)-brass observed in cyclic saturation at a plastic strain amplitude of 0.50% by means of single cycles at the respective lower amplitude](image)

The results obtained in the test with interrupting single cycles indicate that the assumptions of Masing are incompatible to the real processes which govern the cyclic plastic deformation of planar-slip type materials. Figure 13 tries to provide an explanation for this incompatibility. A basis requirement of the simple multi-component models is that there is no interaction between the

![Fig.13. Interpretation of the non-Masing behaviour of planar-slip materials](image)
elementary volumes in the sense that one element affects the deformation characteristics of another. In planar-slip materials the plastic deformation can best be described by a serial arrangement of elements which represent the slip planes. The plastic deformation starts on those planes, which are easiest activated, and spreads out on additional planes, if hardening leads to sufficiently high stresses. Activation of few planes would however mean in a polycrystal that kinks are formed at the surrounding grain boundaries (Fig.13a). Since grain boundaries in solids are normally robust, the material will homogenize deformation (Fig.13b), in order to avoid grain boundary rupture. Hence, additional stresses are produced, which cause a fine spacing of the activated slip planes. These additional forces violate the assumptions of the simple multi-component models.

4.4. Change of Slip Character with Temperature

The results presented so far show that two requirements must be fulfilled to enable a reasonable application of simple multi-component models for the calculation of the stress-strain path of cyclic loading:

- The microstructure of the material must be stabilized.
- The dislocation glide must be of wavy-slip type.

An elegant way to prove the direct correlation of slip character and model assumption is to use a material which allows to establish the slip character by means of the variation of a test parameter. This can be done by using the effect of temperature on the cyclic deformation behaviour of materials which show dynamic strain aging (DSA).

DSA is a phenomenon which occurs in most alloys and which is best known and thoroughly studied for steels [13,14]. Simply speaking, the interaction of lattice defects such as interstitials, vacancies or clusters of such defects with moving dislocations leads to an additional obstruction giving rise to an increased strength. DSA occurs in an intermediate temperature range, where the diffusion of the relevant defects occurs at a similar velocity as the dislocation movement. A consequence of DSA is a planar-type dislocation arrangement that results from the restriction of the dislocation mobility.

Figure 14 compares the different types of dislocation arrangement (left column) observed in an austenitic stainless steel of type 304L with the corresponding hysteresis loops obtained in ISTs and plotted in a Masing representation (right column). Three temperatures are considered: room temperature, 400°C and 650°C. DSA occurs in this material in a rather broad intermediate temperature interval and is strongly pronounced at about 400°C. The dislocation arrangement at this temperature shows a homogeneous distribution and resembles the structure found in planar-slip type metals. At room temperature, wavy slip prevails leading to the formation of a cell structure. At high temperature (e.g., 650°C) also a wavy-type arrangement is established. Here,
the cell structure looks rather orderly and the cell walls are thin (transition to subgrain structure).

Fig.14. Chance of dislocation slip character with temperature and corresponding change in cyclic stress-strain response of an austenitic stainless steel (a: room temperature, b: 400 °C, c: 650 °C)
The change in slip character with increasing temperature from wavy to planar and back to wavy is connected with the expected change from Masing behaviour to Non-Masing behaviour and back to Masing behaviour as can convincingly be seen from the course of the ascending hysteresis loop branches.

4. EXAMPLES OF APPLICATION

The IST can be considered as a special case of a variable-amplitude loading which usually occurs under service conditions. It can be expected from the results reported above that a random loading should also lead to a stabilized microstructure, if the amplitude is changed sufficiently fast and often. Hence, in the case of a wavy-slip material, the stress-strain path can be calculated by applying Masing’s model, if the sequence and the values of the load reversal points are known. This calculation is of technical interest, because the application of advanced fatigue life prediction methods requires often the detailed knowledge of the stress-strain path.

![Fig.15. Probability density function of the microscopic flow stresses in polycrystalline copper under Incremental Step Test loading](image)

The calculation can be carried out easily using Masing’s model (or another multi-component model) in a discrete fashion. In other words, it is assumed that the material which should be described consists of a relatively small number of elements. Then the model can be transferred into a simple computer program. The following examples are based on a discrete model which consists of 33 elements in parallel arrangement.

In a first step, the probability density function $f_p(\sigma_0)$ must be determined by
applying Eq.(9) to a single branch of a hysteresis loop which was recorded under steady state conditions. Figure 15 shows the resulting $f_p(\sigma_i)$ from the evaluation of an ascending branch of an IST hysteresis loop of copper. Due to the double differentiation a rather large scatter occurs in the horizontal part of the hysteresis. Taking the additional requirement into account that the area below the function must be 1, the course of the function can be defined reasonably. Then a discrete distribution can be derived from $f_p(\sigma_i)$ assigning a yield stress and an area fraction to each element. Finally, the entire stress-strain path can be calculated based on these elementary values, if the sequence and the strain values (or, if pertinent, the stress values) of the load reversal points are known.

The excellent agreement between experimentally determined and calculated values is shown in Fig.16 which contains the stress-strain course from a half block performed on copper (solid line) in comparison with the calculation result (dotted line). Differences are hardly appreciable.

It has been mentioned before without providing any evidence that the calculation procedures based on multi-component models implicitly take the memory of prior load history into account. Figure 17 shows the result of a stress-strain calculation which contains small strain changes within a large cycle. During each reloading following a small unloading (see the path connecting point 1 with point 3 for example) a small enclosed hysteresis loop is formed, closed and forgotten. The continuation of the prior stress-strain path
after the closure of the small loop (e.g., at point 2) takes place automatically and reproduces the memory rule.

![Diagram of stress-strain path calculation including load drops and reloadings](image)

Fig.17. Stress-strain path calculation including load drops and reloadings

According to the basic concept of multi-component models, the memory of a solid material results from the locally varying flow stresses which give rise to an internal stress distribution. The prior load history is stored in this stress distribution.

5. CONCLUDING REMARKS

The main results presented in this paper can briefly be summarized as follows.

- Multi-component models can explain basic phenomena of the cyclic stress-strain behaviour.
- The physical basis of such models is the locally different microscopic flow stresses.
- A statistical treatment provides the necessary equations which can be applied to a calculation of the cyclic stress-strain path.
- This calculation on the basis of a flow stress distribution is meaningful only, if wavy slip prevails in the material considered and if a stabilized microstructure has been established.
- In all other cases, the changes of the flow stress distribution must be taken into account in a suitable way.
REFERENCES