Geometrically exact beam theory
S.R. Eugster, C. Hesch, P. Betsch and Ch. Glocker

Introduction

Cosserat beam [1]
- Nonlinear beam finite elements
- Interpolation of the director field
- Does not rely on rotational degrees of freedom
- Frame indifference and conservation of angular momentum

Director-based theory in skew coordinates [2]
- Formulation in convected coordinates
- Accounts for the lack of orthonormality of the discrete director frame
- Improves dramatically the numerical performance

Geometrically exact beam – Kinematical assumptions

- Reference and current configuration

\[ x(R, s, t) = \varphi(s, t) + \theta \cdot \delta d(s, t) \]
where \( R(s, t) \in SO(3) \) such that
\[ \delta d(s) = R(s, t) \delta s, \quad \text{with} \quad R = d_k \otimes e^k \]
- Skew-symmetric effective curvature \( k \)
\[ (d_k)_s = \delta d_k = k \times d_k, \quad \text{with} \quad k = R_s, R^{-1} = d_{kn} \otimes d^k \]

Virtual work and director formulation

- Virtual work
\[ LW = \int \{ \varphi' \cdot (\dot{A}_k \varphi + q - \pi - n_s) + \delta \varphi \cdot (q \times \varphi + k \cdot n - m + \varphi_s - n) \} \mathrm{d} s \]
\[ + (\delta \varphi \cdot (\varphi - n) + \delta \varphi \cdot (\pi - m)) \}\right|_{0} = 0 \quad \forall \varphi, \delta \varphi, \pi, \theta \]
- Constitutive law
\[ W(\gamma, \beta) = \frac{1}{2} e^{1/2}\gamma (D_1)^{-1/2} \gamma + \frac{1}{2} e^{1/2}\beta (D_2)^{-1/2} \beta \]
with
\[ [D_1] = \text{Diag}(G_{11}, G_{12}, E_{11}) \quad \text{and} \quad [D_2] = \text{Diag}(E_{11}, E_{12}, G_{12}) \]
- Variational formulation of the geometrically exact beam
\[ LW = \int \{ \varphi' \cdot \left[ \delta A_{k} + \varphi \delta d_{k} - \pi \right] + \delta \varphi \cdot \left[ \delta (\varphi' \cdot d_{k}) + \varphi' \delta d_{k} - \delta \varphi \right] - \frac{1}{2} \delta \varphi \cdot \left[ \delta A_{k} + \varphi \delta d_{k} - \pi \right] \mathrm{d} s \]
\[ + \frac{1}{2} e^{1/2}\beta (D_2)^{-1/2} \beta \}\right|_{0} = 0 \quad \forall \varphi, \theta, \pi \]

Finite element formulation

- Galerkin type approach
\[ \varphi'(s, t) - \frac{\varphi'(s, t)}{h} \cdot \sum_{m} N_{m} \varphi_{m}(t) = \sum_{m} N_{m} \varphi_{m} \cdot \theta \cdot \delta d_{m}(s, t) \]
\[ \delta \varphi'(s, t) = \sum_{m} N_{m} \varphi_{m}(s, t) - \sum_{m} N_{m} \varphi_{m} \cdot \theta \cdot \delta d_{m}(s, t) \]
- Virtual work of the contact force
\[ LW_{inh} = \delta \varphi(s) \cdot \int_{s_{1}}^{s_{2}} N_{1} \cdot \delta s \]
\[ + \sum_{m} \int_{s_{1}}^{s_{2}} \left[ \left( a_{m} \cdot \delta h_{m} \right) N_{1} \cdot \delta \varphi_{m} - \left( \delta \varphi_{m} \cdot a_{m} \cdot N_{1} \right) \right] \mathrm{d} s \]
- Virtual work of the contact torques
\[ LW_{inh} = \frac{1}{2} \int_{s_{1}}^{s_{2}} \left[ \left( a_{m} \cdot \delta h_{m} \right) N_{1} \cdot \delta \varphi_{m} - \left( \delta \varphi_{m} \cdot a_{m} \cdot N_{1} \right) \right] \mathrm{d} s \]

Numerical example

Beam with slope discontinuity

Errors in tip displacements versus h-refinement.

References

- P. Betsch and P. Steinmann
Frame-indifferent beam finite elements based upon the geometrically exact beam theory.
- S.R. Eugster, C. Hesch, P. Betsch and Ch. Glocker
Director-based beam finite elements relying on the geometrically exact beam theory formulated in skew coordinates