Adaptive Mixed Finite Element Methods for Reissner-Mindlin Plates

Mechanical Model: Elastic Plate
- Plane domain \( \Omega \subset \mathbb{R}^2 \) with thickness \( t \ll \text{diam} \Omega \)
- Bounded by \( f = f(x, y) \) transversal to its midplane
- Elastic material law \( \mathcal{C} = \frac{1}{2} 
abla \sigma = c + \frac{1}{2} f(y, z) (\nu, \lambda \text{Lamé constants}, k \text{ shear correction factor}) \)

Reissner-Mindlin theory: small plate deformations, described by the displacement \( w \) and the rotations \( \theta = (\theta_x, \theta_y) \) with linear strain \( \varepsilon(\theta) = \text{sym}(\nabla \theta) \) and

\[
F(\varepsilon) = \int_{\Omega} \left[ \frac{1}{2} \varepsilon \sigma : \varepsilon + \mu \text{div} \varepsilon \right] \, dx
\]

Shear locking issue: \( t \to 0 \)

Mixed Formulation
- Introduce a linear constraint \( \gamma \geq \left( t^2 - \alpha \right) |\nabla \theta| = 0 \) and establish a mixed variational form [Arnold et al., Arnold, Brezzi et al.]
- Parameter \( a \) is bounded by \( 0 \leq a < t^{\alpha} \), including a classical mixed model with \( a = 0 \)
- Here \( a = a(z) \) is a possibly discontinuous function which may vary with \( z \in \Omega \)

With bilinear forms
\[
\begin{align*}
\alpha(\theta, \mu, \nu) &= \int_{\Omega} \varepsilon(\theta) : \varepsilon(\mu) \, dx \\
b(\theta, w) &= \int_{\Omega} \left( \nabla \theta : \nabla w + \theta \nabla w - \nabla w : \nabla \theta \right) \, dx \\
c(\nu, \gamma) &= \int_{\Omega} \beta |\nabla \gamma| - g dx \\
\beta &= \beta \geq 1/t^2 - \alpha
\end{align*}
\]

the continuous problem reads: \( \text{Find } (w, \theta, \gamma) \in H^1_0(\Omega) \times H^2_0(\Omega) \times L^2(\Omega) \) such that
\[
\begin{align*}
\alpha(\theta, \mu, \nu) + b(\theta, w) + c(\nu, \gamma) &= 0 \\
b(\theta, w) &= 0
\end{align*}
\]

for all \( (\mu, \nu, \gamma) \in H^1_0(\Omega) \times H^2_0(\Omega) \times L^2(\Omega) \).

Finite Element Discretisation
- Finite element mesh \( T \) is a regular partition of \( \Omega \) into closed triangles \( T_1, T_2, \ldots, T_k \)
- Discrete spaces \( V_h \times W_h \times Z_h \subset H^1_0(\Omega) \times H^2_0(\Omega) \times L^2(\Omega) \) consist of \( T \)-dependent polynomials of total degree \( \leq k \)
- \( P_k(T) \) \( \left( u \in L^2(\Omega), \int_{T} u = 0 \right) \)
- \( B_k(T) \) \( \left( u \in P_k(T), \text{div} u = 0 \right) \)
- \( T \rightarrow B_k(T) \neq 0 \)

A posteriori Error Estimation
- For each element \( T \in \mathcal{T} \) we define

\[
\eta_T^2 := h_T^p \int_{T} \left( |w - w_h| + |\nabla w - \nabla w_h| + |\text{div}(\nabla w) - \text{div}(\nabla w_h)| \right) \\
+ h_T \int_{T} \left| \nabla w_h - D y \right| + \int_{T} \left( |\theta - \theta_h| + |\nabla \theta - \nabla \theta_h| \right)
\]

- Problem: Reduced error estimator \( \eta \) can be computed elementwise (once a discrete solution is known) and work as error indicators in an automatic mesh refinement algorithms.

\[
\eta = \left( \sum_{T \in \mathcal{T}} \eta_T^2 \right)^{1/2}
\]

Adaptive Algorithm
- Stabilize mesh by \( h_0 \)
- Solve discrete problem with respect to \( h_0 \) with \( N \) degrees of freedom.
- Compute error estimator \( \eta \) for all \( T \in \mathcal{T} \)
- Compute error bound \( \eta_T^2 := \left( \sum_{T \in \mathcal{T}} \eta_T^2 \right)^{1/2} \) and terminate or go to (c).
- Mark element \( T \) red if \( \eta^2_T > \gamma \max_{T \in \mathcal{T}} \eta^2_T \).
- Refine the mesh to avoid hanging nodes, update mesh \( T \) and go to (b).

Example: Rectangular Sheet Metal

Energy Error
- Instead of computing a reference solution on a very fine mesh (and then providing a lot of data for the public) we compute only one (problem depending) constant \( C \) which allows a representation of natural energy:

\[
c_0 := \left( u^T \sigma(u - u_h) \right)^{1/2} \sqrt{2} \theta - \nabla (w - w_h) \right)
\]

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