Adaptive Mixed Finite Element Methods for Reissner-Mindlin Plates

Dr. Kerstin Weinberg

Graduiertenkolleg 357: Effiziente Algorithmen und Mehrskalenmethoden
Institut für Mathematik - Medizinische Universität zu Lübeck
http://www.math.mu-luebeck.de/weinberg/

Mechanical Model: Elastic Plate

- plane domain $\Omega \in \mathbb{R}^2$ with thickness $t \ll \operatorname{diam}\Omega$
- loaded by $f = f(x, y)$ transversale to its midplane
- elastic material law $\mathbb{C}\varepsilon = \frac{1}{12}\frac{\lambda}{\mu k}\operatorname{tr}\varepsilon I + \frac{1}{6k}\varepsilon$ (μ, λ Lamè–constants, k shear correction factor)

Reissner-Mindlin theory: small plate deformations, described by the displacement w and the rotations $\vartheta = (\vartheta_x, \vartheta_y)$ with linear strain $\varepsilon(\vartheta) = \operatorname{sym}(\nabla\vartheta)$ and

$$t^2 \operatorname{div}\left(\frac{\lambda}{12} I \operatorname{div}\vartheta + \frac{\mu}{6}\varepsilon(\vartheta)\right) + \mu(\nabla w - \vartheta) = 0$$
$$\mu \operatorname{div}(\nabla w - \vartheta) + f = 0$$

problem: *shear locking* $t \to 0$

Mixed Formulation

- introduce a linear constraint $\gamma := (t^{-2} - \alpha)(\nabla w - \vartheta)$ and establish a mixed variational form [Arnold et. al., Braess, Brezzi et. al.]
- parameter α is bounded by $0 \le \alpha < t^{-2}$, including a classical mixed model with $\alpha = 0$
- here $\alpha = \alpha(x)$ is a possibly discontinuous function which may vary with $x \in \Omega$

With bilinear forms

$$\begin{aligned} a(\vartheta, w; \varphi, v) &:= \int_\Omega \varepsilon(\vartheta) : \mathbb{C}\varepsilon(\varphi)\, dx \\ &\quad + \int_\Omega \alpha\,(\vartheta - \nabla w)\cdot(\varphi - \nabla v)\, dx \\ b(\vartheta, w; \eta) &:= \int_\Omega (\vartheta - \nabla w)\cdot \eta\, dx \\ c(\gamma; \eta) &:= \int_\Omega \beta\, \gamma \cdot \eta\, dx \qquad \beta := 1/(t^{-2} - \alpha) \end{aligned}$$

the **continuous problem** reads: *Find* $(w, \vartheta, \gamma) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \times L^2(\Omega)^2$ *such that*

$$\begin{aligned} a(w, \vartheta; v, \varphi) + b(v, \varphi; \gamma) &= (f; v) \\ b(w, \vartheta; \eta) - c(\gamma; \eta) &= 0 \end{aligned}$$

for all $(v, \varphi, \eta) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \times L^2(\Omega)^2$.

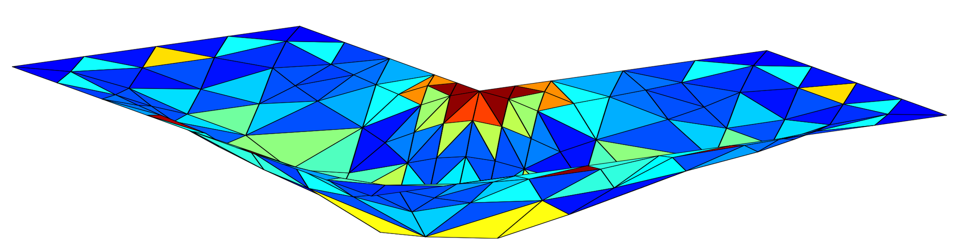
Finite Element Discretisation

- finite element mesh $\mathcal{T}$ is a regular partition of Ω into closed triangels $T_1, T_2, \ldots, T_n$
- discrete subspaces $V_h \times W_h \times \Gamma_h$ of $H_0^1(\Omega) \times H_0^1(\Omega)^2 \times L^2(\Omega)^2$ consist of $\mathcal{T}$-piecewise polynomials of total degree $\le k$

$$\begin{aligned} \mathcal{P}_k(\mathcal{T}) &:= \{u \in L^2(\Omega) \mid \forall T \in \mathcal{T},\ u|_T \in \mathcal{P}_k(T)\} \\ \mathcal{B}_3(\mathcal{T}) &:= \{u|_T \in \mathcal{P}_3(T) \text{ and } u = 0 \text{ on } \partial T\} \end{aligned}$$

- here $\Gamma_h := \mathcal{P}_0(\mathcal{T})^2$ and e.g.

$$\begin{aligned} V_h \times W_h &:= \mathcal{P}_2(\mathcal{T}) \times (\mathcal{P}_1(\mathcal{T}) \oplus \mathcal{B}_3(\mathcal{T}))^2 \\ V_h \times W_h &:= \mathcal{P}_2(\mathcal{T}) \times \mathcal{P}_2(\mathcal{T})^2 \end{aligned}$$



L-shaped plate, all side clamped and uniformly loaded

The **discrete problem** reads: *Find* $(w_h, \vartheta_h, \gamma_h) \in V_h \times W_h \times \Gamma_h$ *such that*

$$\begin{aligned} a(w_h, \vartheta_h; v_h, \varphi_h) + b(v_h, \varphi_h; \gamma_h) &= (g; v_h) \\ b(w_h, \vartheta_h; \eta_h) - c(\gamma_h; \eta_h) &= 0 \end{aligned}$$

for all $(v_h, \varphi_h, \eta_h) \in V_h \times W_h \times \Gamma_h$.

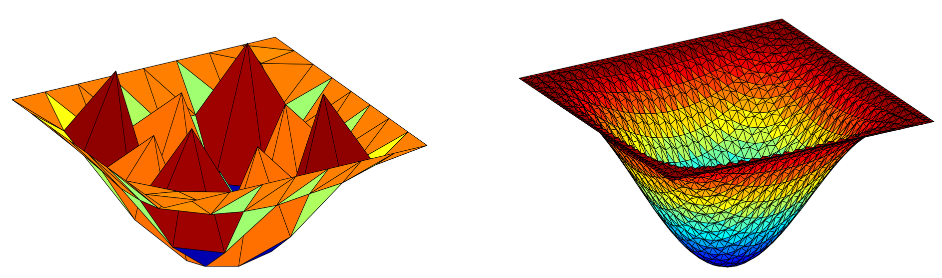
- a priori error estimation: With (h, t)-independent positive constant c there holds

$$\begin{aligned} &\| \vartheta - \vartheta_h \|_{H_0^1(\Omega)} + \| w - w_h \|_{H_0^1(\Omega)} + \| \gamma - \gamma_h \|_{H^{-1}(\operatorname{div})\cap t\cdot L^2} \\ &\le c\Big(\inf_{\phi_h \in V_h} \| \vartheta - \phi_h \|_{H_0^1(\Omega)} + \inf_{v_h \in W_h} \| w - v_h \|_{H_0^1(\Omega)} \\ &\qquad + \inf_{\eta_h \in \Gamma_h} \| \gamma - \eta_h \|_{H^{-1}(\operatorname{div})\cap t\cdot L^2} \Big). \end{aligned}$$

- discretisations are locking free and asymptotically stable with mesh size $h \to 0$ [Arnold & Brezzi, Lovadina & Boffi]

Preasymptotic Performance

To assess whether the asymptotical predictions are observed for applicable, reasonable sized (but for the mathematical analysis possibly too coarse) meshes too, needs numerical investigation.



Quadratic plate uniformly loaded: displacement function computed with $\alpha = 1$

Parameter α influences the approximation quality essentially [Chapelle & Stenberg, Carstensen & W.].

- if α is a small number (what even may $\alpha = 1$ as in [Braess]) the system shows spurious modes
- large numbers of α let the system becomes stiff
- *optimal convergence:* mesh adapted parameter α computed by $\alpha = \alpha(T) = 1/(h_T^2 + t^2)$

Energy Error

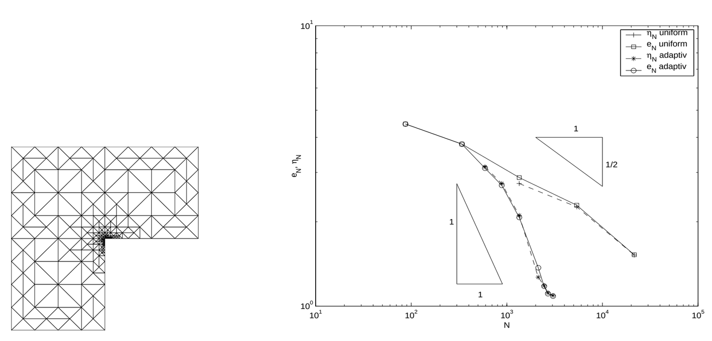
Instead of computing a reference solution on a very fine mesh (and then providing a lot of data for the public) we compute only one (problem depending) constant C which allows a representation of natural energy error

$$e_h := \left(\|\mathbb{C}^{1/2}\varepsilon(\vartheta - \vartheta_h)\,\|^2 + t^{-2}\|\,\vartheta - \vartheta_h - \nabla(w - w_h)\,\|^2\right)^{1/2}$$

what equals the square root of

$$\begin{aligned} C\ -\ &\|\mathbb{C}^{1/2}\varepsilon(\vartheta_h)\,\|^2 + t^{-2}\|\,\nabla w_h - \vartheta_h\,\|^2 \\ &-2\big((t^{-2} - \alpha)(\nabla w_h - \vartheta_h) - \gamma_h; \nabla w_h - \vartheta_h\big) \end{aligned}$$

with $C := (f; w)$. Use a finite element calculation on a very fine mesh or an extrapolation technique to provide constant C.



Relative energy error $e_N = e_h/\sqrt{C}$ versus the degrees of freedom with uniform and adaptive mesh refinement

A posteriori Error Estimation

For each finite element $T \in \mathcal{T}$ we define

$$\begin{aligned} \eta_T^2 := h_T^2 \int_T \Big(&|f - \operatorname{div}(\gamma_h + \alpha(\vartheta_h - \nabla w_h))|^2 \\ &+ |\gamma_h + \alpha(\vartheta_h - \nabla w_h) - \operatorname{div}\mathbb{C}\varepsilon(\vartheta_h)|^2 \Big) dx \\ &+ h_T/t \int_T |\nabla\vartheta_h - D^2 w_h|^2 dx \\ + \sum_{E\in\mathcal{E},\, E\subset\partial T} h_E \int_E \Big(&|[\gamma_h + \alpha(\vartheta_h - \nabla w_h)]\cdot n_E|^2 \\ &+ |[\mathbb{C}\varepsilon(\vartheta_h)]\cdot n_E|^2 \Big)\, ds. \end{aligned}$$

Local contributions η_T to error estimator η can be computed elementwise (once a discrete solution is known) and work as *error indicators* in automatic mesh refining algorithms.

$$\eta = \Big(\sum_{T\in\mathcal{T}} \eta_T^2\Big)^{1/2}$$

- reliability of a posteriori error estimator η: With (h, t)-independent positive constant c there holds

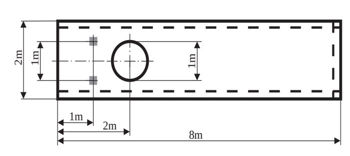
$$\begin{aligned} \eta \ge c\big(&\| \vartheta - \vartheta_h \|_{H_0^1(\Omega)} + \| w - w_h \|_{H_0^1(\Omega)} \\ &+ \| \gamma - \gamma_h + \alpha(\vartheta - \vartheta_h - \nabla(w - w_h)) \|_{H^{-1}(\operatorname{div})\cap t\cdot L^2}\big). \end{aligned}$$

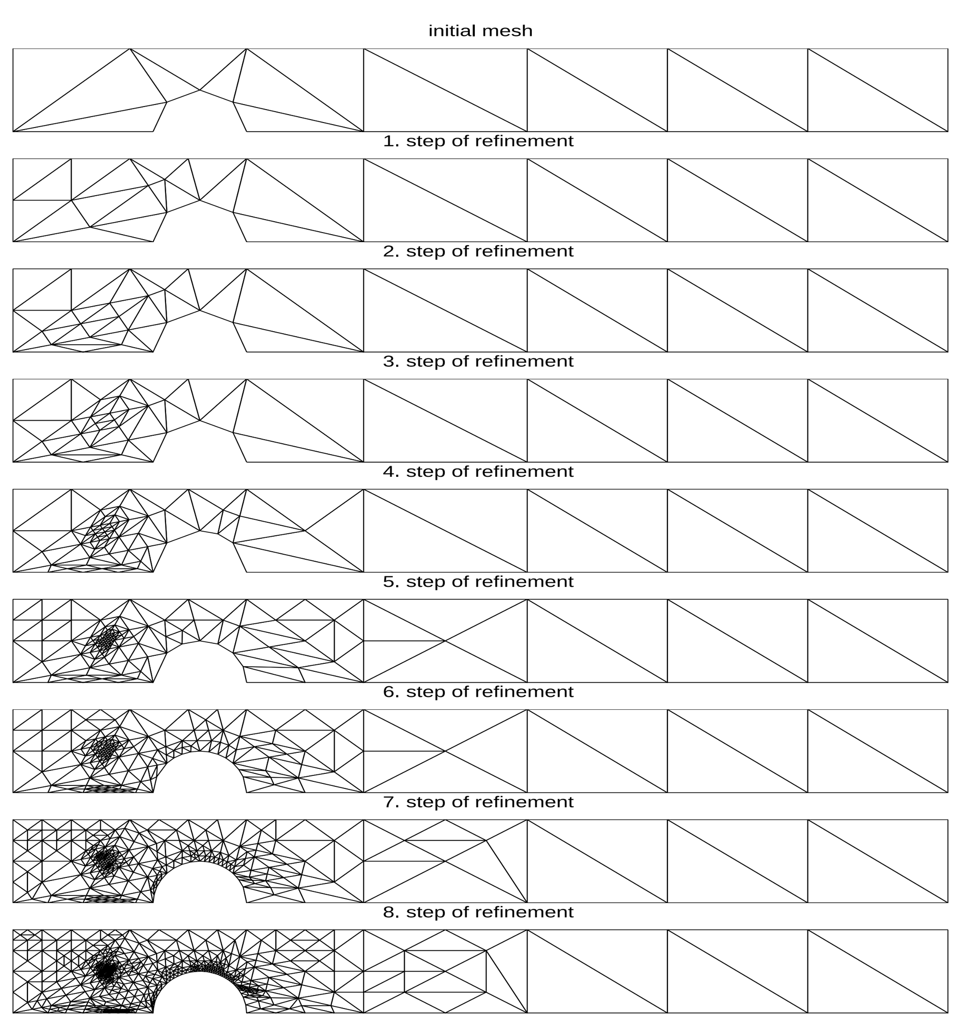
Adaptive Algorithm

(a) Start with coarse mesh $\mathcal{T}_0$.
(b) Solve discrete problem with respect to $\mathcal{T}_k$ with N degrees of freedom.
(c) Compute η_T for all $T \in \mathcal{T}_k$.
(d) Compute error bound $\eta_N := \left(\sum_{T\in\mathcal{T}_k} \eta_T^2\right)^{1/2}$ and terminate or go to (e).
(e) Mark element T red iff $\eta_T \ge \frac{1}{2}\max_{T'\in\mathcal{T}_k} \eta_{T'}$.
(f) Red-green-blue-refinement to avoid hanging nodes, update mesh $\mathcal{T}_k$ and go to (b).

Example: Rectangular Sheet Metal

hard simply supported on three sides and loaded by two stamps of 0.1 m × 0.1 m (e.g., caused by fork-lift trucks)





Meshes generated by Adaptive Algorithm