An Adaptive pNh-Technique for Global–Local Finite Element Analysis

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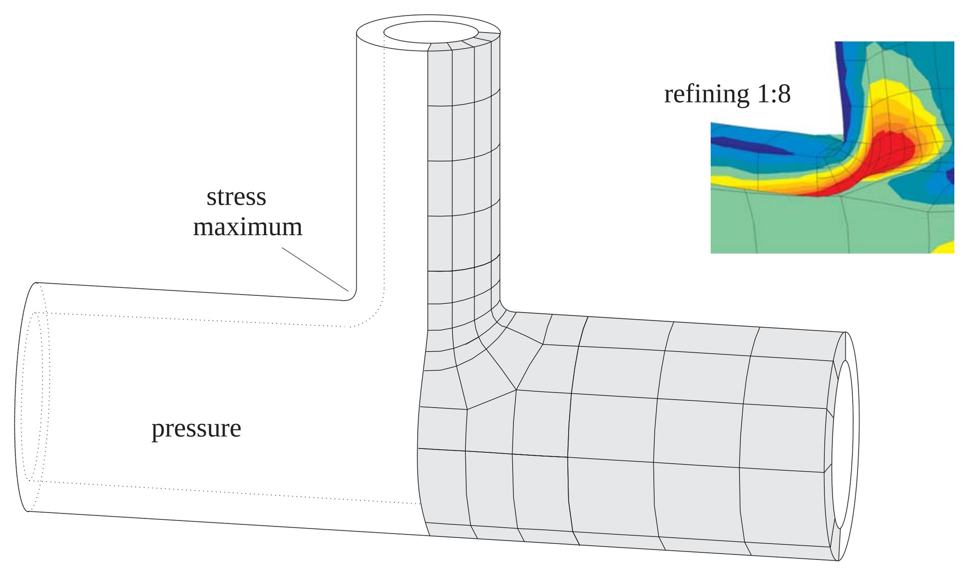
Motivation

- a finite element analysis shall compute engineering problems as accurate as possible at reasonable cost
- h-version: reduce mesh size $h \to 0$
- p-version: rise polynomial degree $p \to \infty$

Elasticity: Approximate the displacement function $\mathbf{u}(\mathbf{x})$ of the domain $\Omega(\mathbf{x}) \in \mathbb{R}^3$ and then the stresses $\sigma(\mathbf{u}) = \mathbb{C}\varepsilon(\mathbf{u})$ by solving the variational problem:

$$\int_\Omega \varepsilon(\mathbf{u}) : \mathbb{C}\varepsilon(\mathbf{u})\, dx - \int_\Omega \mathbf{f} \cdot \mathbf{u}\, dx \to min$$

with linear Green strain $\varepsilon(\mathbf{u}) = \mathrm{sym}(\nabla \mathbf{u})$, material tensor $\mathbb{C}\varepsilon = \frac{E\nu}{(1+\nu)(1-2\nu)} \mathrm{tr}\, \varepsilon I + \frac{E}{1+\nu}\varepsilon$ (Hooke's law) and load function $\mathbf{f}(\mathbf{x})$.

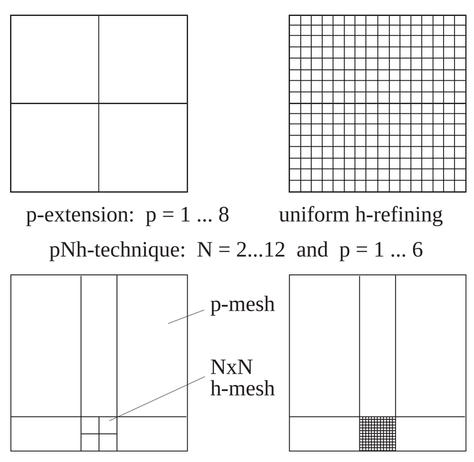


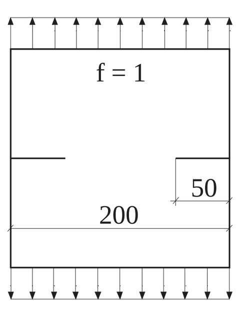
Pipe Branching

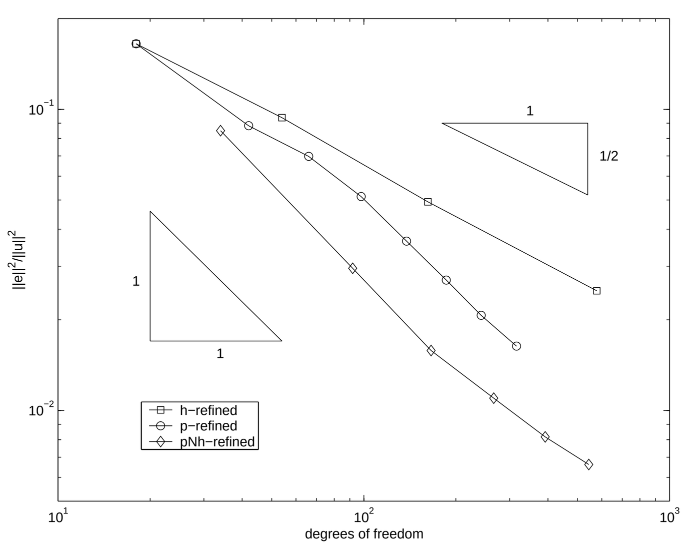
Typically the solution is expected to be a smooth function in wide parts of Ω and may efficiently be approximated with few large high order p-version elements, but some parts of Ω need a fine h-mesh to compute high gradients or singularities (e.g. owing to cracks, geometrical details, material modifications, unknown contact areas, ...).
Engineering problems require a flexible interface for coupling p- and h-refined meshes!

Discretisation of a Cracked Plate

- analytical solution $\| \mathbf{u} \|_E^2 = 1.46 \cdot 10^4$
- stress singularity at crack tip
- plane stress state



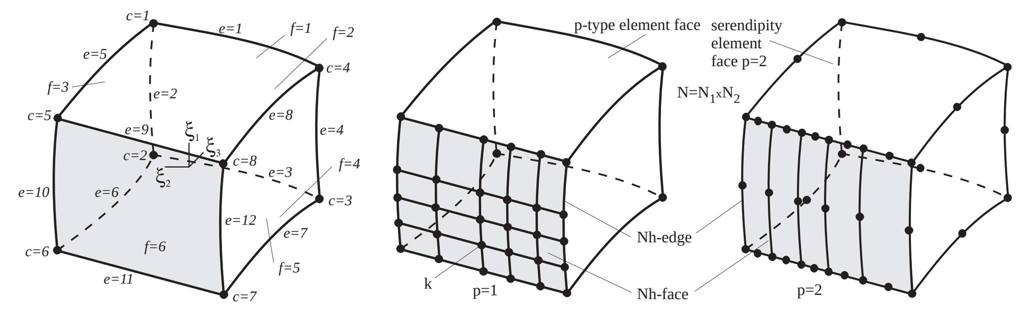




Convergence with Mesh Refining

The pNh-Transition–Element Technique

The pNh-transition elements are special p-version elements with usual polynomial shape functions and with an (arbitrary) number N of piecewise defined h-version polynomials at one or more sides or faces of the elements.



e.g.: The hierarchically defined discrete displacement function $\tilde{\mathbf{u}}(\boldsymbol{\xi})$, $\xi_i \in [-1,1]_{i=1,2,3}$ of a hexaedron element reads:

$$\tilde{\mathbf{u}}(\boldsymbol{\xi}) = \underbrace{\sum_{k=1}^{4} \mathcal{A}_k \hat{\mathbf{u}}_\mathcal{A}}_{\text{p-node-modes}} + \underbrace{\sum_{e=1}^{8}\sum_{\zeta=2}^{p} \mathcal{B}_s^\zeta \hat{\mathbf{u}}_\mathcal{B}}_{\text{p-edge-modes}} + \underbrace{\sum_{f=1}^{5}\sum_{\zeta=2}^{p-2}\sum_{\eta=2}^{p-\zeta} \mathcal{C}_f^{\zeta\eta} \hat{\mathbf{u}}_\mathcal{C}}_{\text{p-face-modes}}$$
$$+ \underbrace{\sum_{\zeta=2}^{p-4}\sum_{\eta=2}^{p-\zeta}\sum_{\iota=2}^{p-\zeta-\eta} \mathcal{D}^{\zeta\eta\iota} \hat{\mathbf{u}}_\mathcal{D}}_{\text{volume-modes}} + \underbrace{\sum_{k_1=1}^{N_1+1}\sum_{k_2=1}^{N_2+1} \mathcal{E}_{k(k_1k_2)} \hat{\mathbf{u}}_\mathcal{E}}_{\text{Nh-modes}}$$

We define with $\gamma_j(\xi_i) = \frac{1}{2}(1+\xi_{ij}\xi_i)$ and normalized integrals of the Legendre–polynomials $P_\zeta(x) = \frac{1}{2^\zeta \zeta!} \cdot \frac{d^\zeta}{dx^\zeta}(x^2-1)^\zeta$

$$\Phi_\zeta(\xi) = \sqrt{\frac{2\zeta-1}{2}} \int_{-1}^{\xi} P_{\zeta-1}(x)\, dx \quad \zeta \geq 2$$

- p-node modes (e.g. c=1):
$$\mathcal{A}_1 = \gamma_1(\xi_1) \cdot \gamma_1(\xi_2) \cdot \gamma_1(\xi_3)$$
- p-edge modes (e.g. e=1):
$$\mathcal{B}_1^\zeta = \gamma_1(\xi_1) \cdot \Phi_\zeta(\xi_2) \cdot \gamma_1(\xi_3)$$
- p-face modes (e.g. f=1):
$$\mathcal{C}_1^{\zeta\eta} = \Phi_\zeta(\xi_1) \cdot \Phi_\eta(\xi_2) \cdot \gamma_1(\xi_3)$$
- p-volume modes:
$$\mathcal{D}^{\zeta\eta\iota} = \Phi_\zeta(\xi_1) \cdot \Phi_\eta(\xi_2) \cdot \Phi_\iota(\xi_3)$$
- Nh-modes (e.g. f=6, node k):
$$\mathcal{E}_k = \phi_k(\xi_1) \cdot \phi_k(\xi_2) \cdot \gamma_7(\xi_3).$$

A piecewise linear Nh–function $\phi_k(\xi)$ reads

$$\phi_k(\xi) = \frac{h_k + |\xi - \xi_{k-1}| - 2|\xi - \xi_k| + |\xi - \xi_{k+1}|}{h_k - \xi_{k-1} + \xi_{k+1}}$$
$$h_k = (\xi_{k-1} - 2\xi_k + \xi_{k+1}) \cdot \mathrm{sgn}(\xi - \xi_k).$$

Consequently the finite element mesh $\mathcal{T}$ is a regular partition of Ω (C_0 continuous, no hanging nodes). Thus we deduce an a posteriori error estimate $\eta = (\sum_{T\in\mathcal{T}} \eta_T^2)^{1/2}$ with residue based contributions of each element $T \in \mathcal{T}$

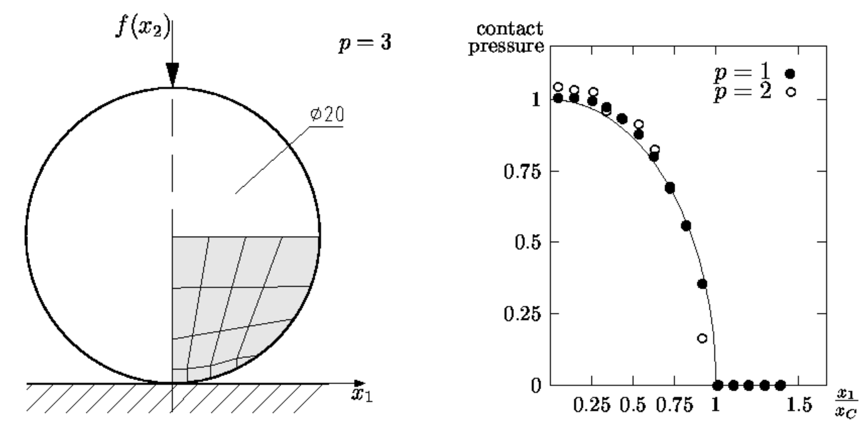
$$\eta_T^2 = \sum_{s=1}^{N} \Big(h_s^2 \int_{T_s} |\mathrm{div}\big(\mathbb{C}\varepsilon(\tilde{\mathbf{u}})\big) - \mathbf{f}|^2\, dx$$
$$+ c h_s \int_{\partial T_s} |[\sigma(\tilde{\mathbf{u}})] \cdot n(\partial T_s)|^2\, ds \Big).$$

The local contributions η_T can work as error indicators in an adaptive mesh refining algorithm.

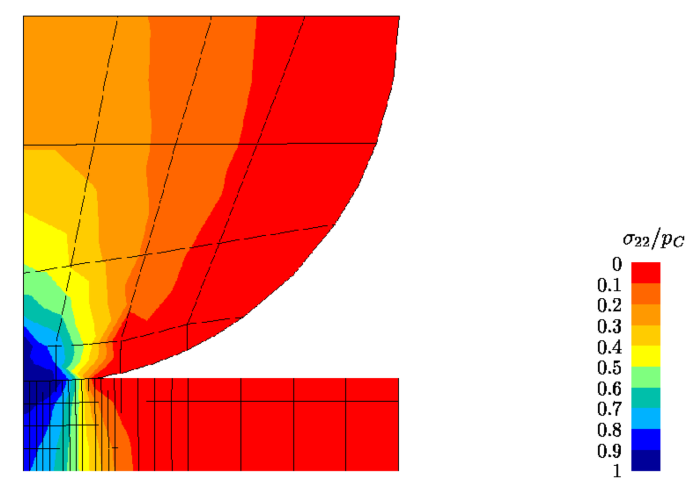
Contact Problems

The calculation of a priori unknown contact areas requires a fine h-mesh and an iterative solution process to meet static and kinematic contact conditions $p_C \leq 0$, $u_A - u_B - u_{gap} = 0$.

- a coarse p-version mesh is not useful
- employing the pNh-technique the p-version is available for general finite elemente contact algorithms

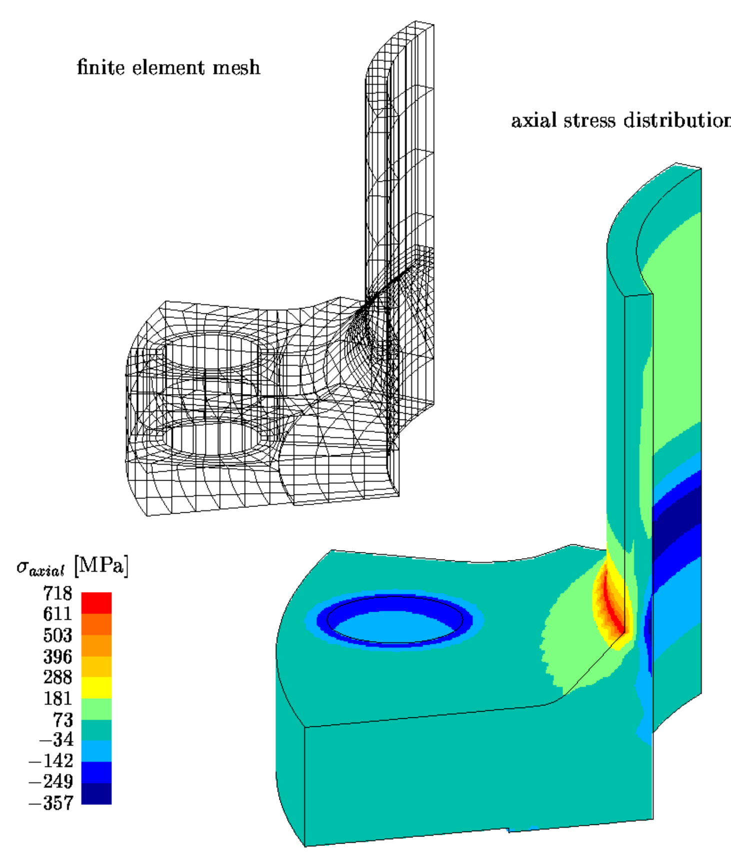


Example: Cylinder on an Elastic Foundation



Example: Calculation of a Flange

- pipe union, loaded with inside pressure and traction
- efficient a priori mesh refining in critical regions



Stress Computation of a $\frac{1}{8}$ Symmetric Sector

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