A geometrically exact membrane model for isotropic foils and fabrics

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The finite-strain-viscoelastic membrane model

The spatial deformation of a thin-walled structure $\phi_s : \omega \times (-\frac{h}{2}, \frac{h}{2}) \to \mathbb{R}^3$ is decomposed into the motion of the (initially plane) midsurface $m : \omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^2$ and of the director (initially) orthogonal to the midsurface,

$$\phi_s(x, y, z) = m(x, y) + z\varrho_m(x, y)\, R(x, y).e_3\,, \tag{1}$$

where $R = \text{polar}(F) \in \text{SO}(3)$ is the orthogonal part of the deformation gradient F with out-of plane component $R(x, y).e_3$. The variable $\varrho_m \in \mathbb{R}$ accounts for a varying thickness, see [1, 2] for details.

Basic idea: introduce an *additional* field of independently evolving viscoelastic rotations $\overline{R} \in \text{SO}(3)$. These rotations $\overline{R}$ are thought of as being physical meaningful but not exact continuum rotations R. With $R_3 \equiv \overline{R}(x, y).e_3$ denoting the corresponding out-of plane component the dimensional reduction of a three-dimensional continuum solid to a geometrically exact membrane model results in a deformation gradient of the form

$$F = (\nabla m | \varrho_m \overline{R}_3), \tag{2}$$

where $\nabla m \in \mathbb{M}^{3\times 3}$ is the deformation gradient of the midsurface with $m_x = (m_{1,x}, m_{2,x}, m_{3,x})^T$, $m_y = (m_{1,y}, m_{2,y}, m_{3,y})^T$.

The problem: find the deformation of the midsurface $m : [0, T] \times \omega \mapsto \mathbb{R}^3$ and the independent local viscoelastic rotation $\overline{R} : [0, T] \times \omega \mapsto \text{SO}(3, \mathbb{R})$ such that

$$\int_\omega h\, W(F, \overline{R})\, \text{d}\omega - \int_\omega \langle f_b, m\rangle\, \text{d}\omega - \int_{\gamma_s} \langle f_s, m\rangle\, \text{ds} \mapsto \min.\,, \tag{3}$$

w.r.t. m at fixed rotation $\overline{R}$. The strain energy density $W(F, \overline{R})$ in (3) is of the form

$$W(F, \overline{R}) = \frac{\mu}{4}\, \|F^T\overline{R} + \overline{R}^T F - 2I\|^2 + \frac{\lambda}{8}\, \text{tr}\, \left(F^T\overline{R} + \overline{R}^T F - 2I\right)^2. \tag{4}$$

Moreover, let $W^{\text{ext}}(m)$ be the linear work of applied external forces with f_b being the resultant body forces and f_s the resultant surface traction and let $g_{\text{d}} : \omega \mapsto \mathbb{R}^3$ denote the prescribed Dirichlet boundary conditions for the membrane,

$$W^{\text{ext}}(m) = \int_\omega \langle f_b, m\rangle\, \text{d}\omega - \int_{\gamma_s} \langle f_s, m\rangle\, \text{ds}\,,$$
$$m_{|\gamma_0}(t, x, y) = g_{\text{d}}(t, x, y) \qquad x, y \in \gamma_0 \subset \partial\omega\,. \tag{5}$$

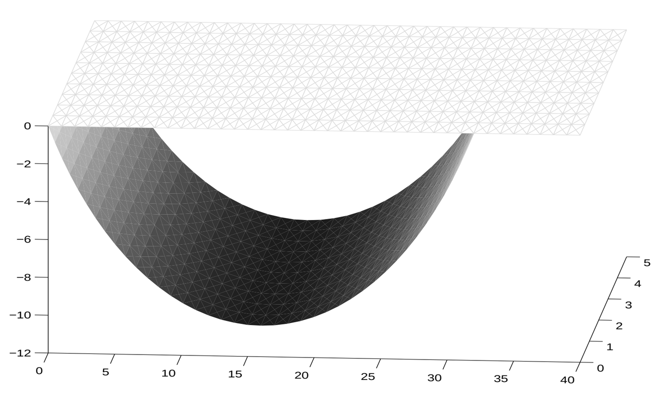
The field of local viscoelastic rotation follows an **evolution equation**

$$\frac{\text{d}}{\text{d}t}\overline{R}(t) = \nu^+ \cdot \text{skew}(B) \cdot \overline{R}(t) \quad \text{with} \quad \nu^+ := \frac{1}{\eta}\nu^+(F, \overline{R}), \quad \text{and } B = F\overline{R}^T. \tag{6}$$

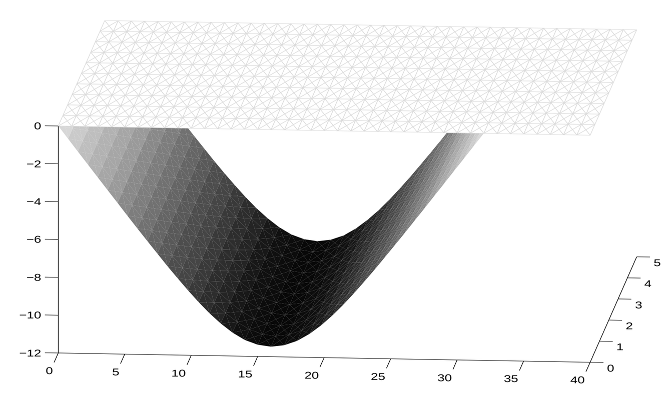
Here $\nu^+ \in \mathbb{R}^+$ represents a scalar valued function introducing an *artificial viscosity* and η plays the role of an *artificial relaxation time* (with units [sec]). The evolution equation (6) and parameter ν^+ are introduced into the model to preserve ellipticity of the force balance. Physically, one may imagine the viscoelastic rotation $\overline{R}$ as *shadowing* the exact continuum rotation in a viscous sense.

Example: rectangular sheet

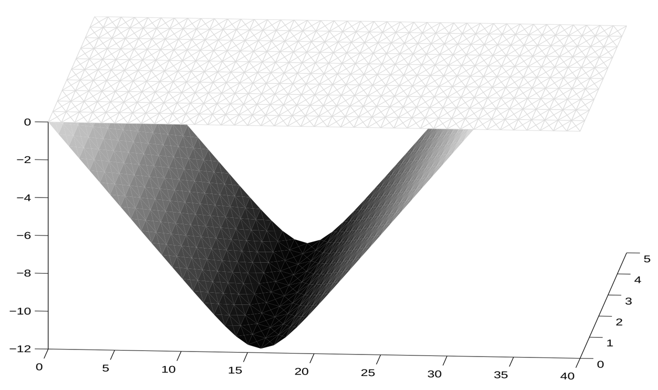
A hard sheet is loaded by dead load and subjected to in-plane displacement of one side. The figures show the initial and the deformed state after different time periods of relaxation:



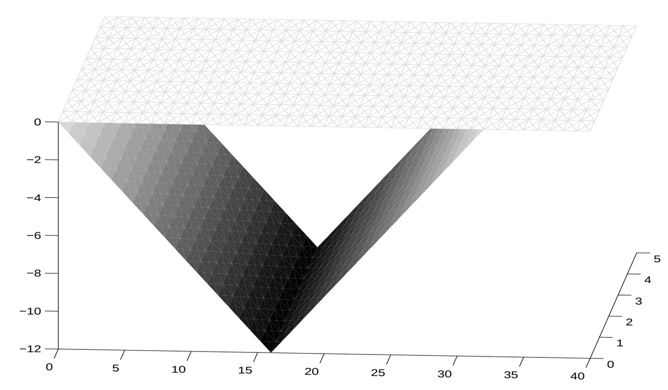
no relaxation time



relaxation time 500s



relaxation time 1000s



relaxation time 5000s

Discretization of the model

We consider a fully implizit time discretized version of model (3). Let $(m^{n-1}, \overline{R}^{n-1})$ be the given solution for the deformation of the midsurface and the rotations at time t_{n-1}. Now, compute the new solution $(m^n, \overline{R}^n) \in \mathbb{V}$ at time t_n such that

$$\int_\omega h\, W(F^n, \overline{R}^n)\, \text{d}\omega - W^{\text{ext,n}}(m^n) \mapsto \min.\,, \tag{7}$$

w.r.t. m^n at fixed $\overline{R}^n$. The current deformation gradient $F^n = F(t_n)$ is

$$F^n = (\nabla m^n | \varrho_m^n \overline{R}_3^n)\,, \tag{8}$$

and the current boundary conditions are

$$m^n_{|\gamma_0}(t_n, x, y) = g_{\text{d}}(t_n, x, y)\,, \qquad x, y \in \gamma_0 \subset \partial\omega\,. \tag{9}$$

The **evolution equation** for the rotations is mapped by a **local exponential update**. This implies that $\overline{R}^n = \overline{R}^n(\nabla m^n)$ solves the following highly nonlinear problem

$$\overline{R}^n = \exp\left(\Delta t\, \nu_n^+\, \text{skew}\left(F^n \overline{R}^{n,T}\right)\right) \cdot \overline{R}^{n-1} \quad \text{with } \nu_n^+ = \frac{1}{\eta}\left(1 + \|\, \text{skew}\, F^n \overline{R}^{n,T}\|\right)^2. \tag{10}$$

By the properties of logarithmic and exponential mapping it can be shown that (10) converges to (6) for the limit $\Delta t \to 0$, see [1].

The **finite element discretization** of problem (7) considers discrete subspaces $\mathbb{V}_\text{h}$ of the continuous solution spaces $\mathbb{V}$ for the membrane's deformation. We employ

$$\mathbb{V}_\text{h} = \mathcal{P}^\text{o}{}_1(\mathcal{T})^3 \times \mathcal{P}_0(\mathcal{T})^{3\times 3}\,, \tag{11}$$

where $\mathcal{P}_k(\mathcal{T})$ denotes the linear space of $\mathcal{T}$-piecewise polynomials of degree $\leq k$, and, $\mathcal{P}^\text{o}{}_k(\mathcal{T})$ are the continuous discrete functions in $\mathcal{P}_k(\mathcal{T})$ with homogeneous boundary values. Thus, the **discrete problem** reads: find the deformation of the midsurface of the membrane and the independent local viscoelastic rotation $(m_\text{h}, \overline{R}_\text{h}) : [0, T] \times \mathbb{V}_\text{h}$ such that,

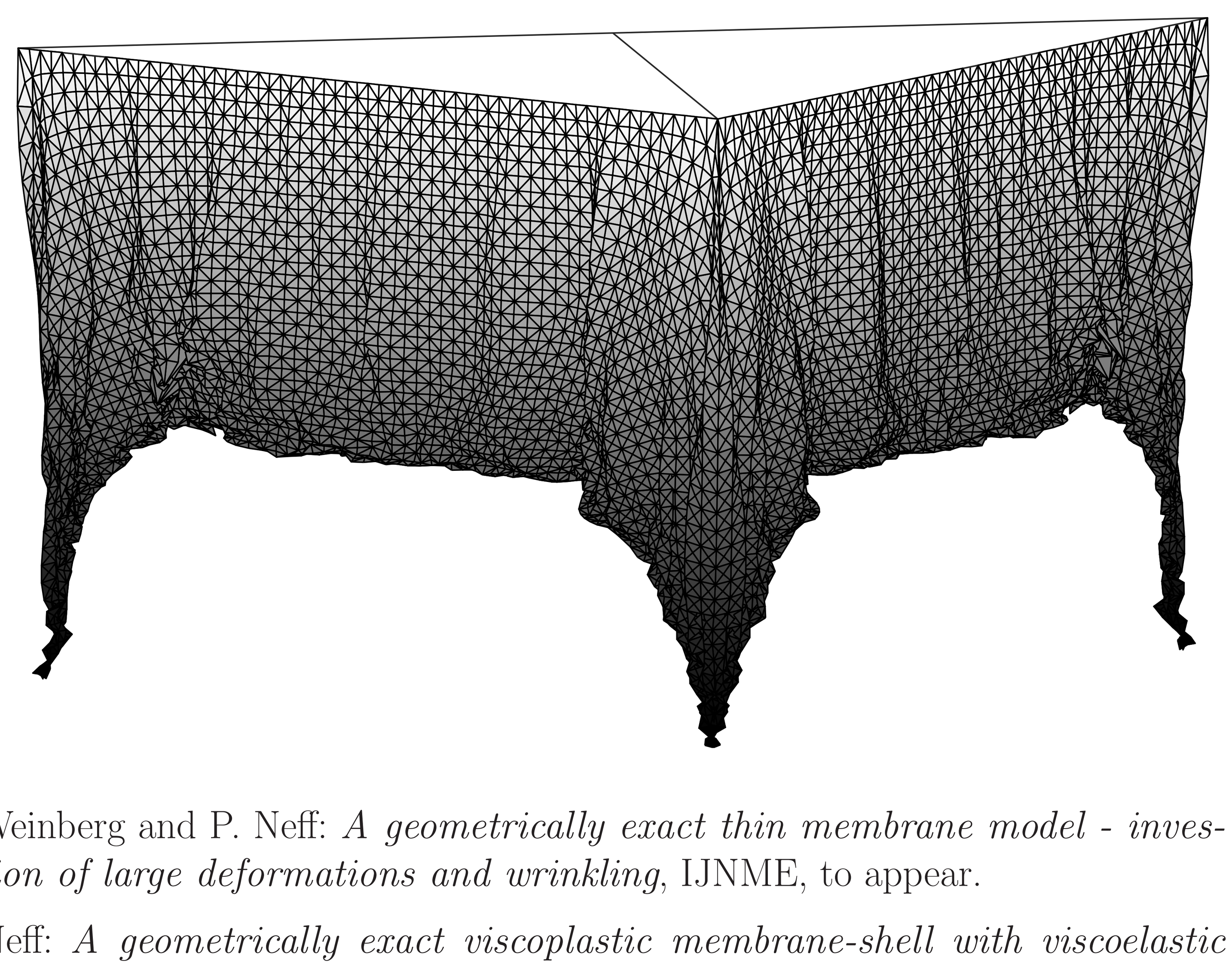
$$\int_\omega h\, W(F(m_\text{h}), \overline{R}_\text{h})\, \text{d}\omega - W^{\text{ext}}(m_\text{h}, \overline{R}_{\text{h}3}) \mapsto \min.\,, \tag{12}$$

w.r.t. m_h at fixed rotation $\overline{R}_\text{h}$ such that R_h satisfies (10).

Example: wrinkling of a thin foil

We apply our model to the problem of a 2×2m elastic foil under pressure load. The foil is 1mm thick, lays on a square obstacle (like a cloths on a table) and only the unsupported part of it can deform. A pressure of $p_0 = 0.75$ MPa acts from above.

Wrinkling of a soft foil (relaxed state):



[**1**] K. Weinberg and P. Neff: *A geometrically exact thin membrane model - investigation of large deformations and wrinkling*, IJNME, to appear.

[**2**] P. Neff: *A geometrically exact viscoplastic membrane-shell with viscoelastic transverse shear resistance avoiding degeneracy in the thin-shell limit. Part I: The viscoelastic membrane-plate.*, ZAMP, **56** (2005), 148–182.