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A brief history of logarithmic strain measures in nonlinear elasticity

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1 Early uniaxial logarithmic laws

Historically, there have been a number of different motivations for the use of logarithmic strain measures in nonlinear elasticity. The use of logarithmic strain measures in nonlinear elasticity theory dates back to the end of the 19$^{\text{th}}$ century.

In 1880, **A. Imbert** proposed a logarithmic stress response function as a model for the *uniaxial* tension of vulcanized rubber [10]. **E. Hartig** applied a similar law to the uniaxial deformation of rubber [4] in 1893. They employed the equation $t = c \cdot \ln(\lambda)$ with some material parameter c to describe the relation between the observed uniaxial elongation λ and the required force t.

However, both of them used a purely **phenomenological approach**: neither Imbert nor Hartig considered a theoretical framework or stated underlying reasons for the use of a logarithmic strain measure. They merely employed the logarithm in order to give an approximation of data obtained through (uniaxial) experiments.

Today, the introduction of logarithmic strain measures to the theory of elasticity is often attributed to **P. Ludwik**, for example by Truesdell [23, p. 254]. The earliest mention of the logarithmic strain by Ludwik appears in his 1909 monograph *Elemente der technologischen Mechanik* [12] on plastic deformations, where Ludwik arrived at the logarithmic strain measure via the integral $\int_{l_0}^{l} \frac{dl}{l} = \log \frac{l}{l_0}$ over the *instantaneous strain* $\frac{dl}{l}$ for uniaxial elongations.

2 The work of Heinrich Hencky

A fully three-dimensional logarithmic elastic law, widely considered to be the first of its kind, was introduced by **Heinrich Hencky** in his 1928 article *Über die Form des Elastizitätsgesetzes bei ideal elastischen Stoffen* [5, 14]. His approach can be summarized as follows: Hencky assumed that a **law of superposition** of the form

$$\sigma(V_1 V_2) = \sigma(V_1) + \sigma(V_2) \qquad (1)$$

holds for all coaxial, i.e. commuting, stretches $V_1, V_2 \in \mathrm{Sym}^+(3)$; here, σ denotes the corresponding Cauchy stress. From this assumption, he deduced a logarithmic law of elasticity. He also gave an explicit motivation for his assumed law of superposition, which he later expanded upon in a 1929 article [6, 14]: referring to Prandtl's distinction between "elastically determinate" and "elastically indeterminate constructs" [17], Hencky assumes that a law of elasticity for an *ideally* elastic body should provide "elastic determinacy to the greatest extent for epistemological reasons" [14, p. 19], a requirement motivated by Dingler [3]. From this he concludes that the multiplicative composition of coaxial stretches must effect the additive composition of the respective Cauchy stresses σ. Hencky correctly claimed that an elastic stress response which satisfies (1) must necessarily be of the form

$$\sigma(V) = 2\,\mu \cdot \mathrm{dev}_3 \log(V) + \kappa \cdot \mathrm{tr}[\log V] \cdot \mathbb{1} \qquad (2)$$

with material parameters $\mu, \kappa > 0$.

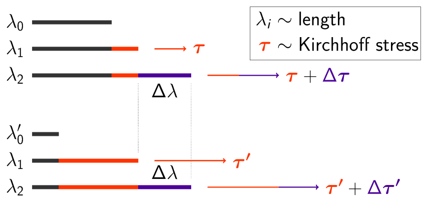
In another 1929 article [7, 14], however, Hencky corrected his statements, proposing then that the law of superposition must hold for the Kirchhoff stress tensor τ instead of the Cauchy stress σ. Although his reasoning for this correction is based on L. Brillouin's suggestion [2] that the Cauchy stress "is not a true tensor of weight 0 but a tensor density" as well as a "lack of group properties for pure deformations in the general case" [14, p. 20], the fact that the stress-stretch relation

$$\tau(V) = 2\,\mu \cdot \mathrm{dev}_3 \log(V) + \kappa \cdot \mathrm{tr}[\log V] \cdot \mathbb{1} \qquad (3)$$

resulting from this new approach with respect to the Kirchhoff stress τ is *hyperelastic* with the corresponding strain energy (cf. [15])

$$W(V) = \mu\,\|\mathrm{dev}_3 \log V\|^2 + \frac{\kappa}{2}[\mathrm{tr}(\log V)]^2 \qquad (4)$$

can be seen as a motivating factor as well, especially since Hencky had shown in his 1928 article that the stress response (2) does not lead to a path-independent energy potential and is therefore *not* hyperelastic.



Darum verlangen wir noch zweitens eine solche Form des Elastizitätsgesetzes, daß beim Aufbringen einer neuen Last auf einen schon belasteten Körper ein Erschließen der alten Last aus den Formänderungen, die beim Aufbringen der Zusatzbelastung eintreten, unmöglich sein soll.

Figure 1: Hencky's law of superposition in the one-dimensional case: regardless of the predeformation and the prior stress, it must hold that $\Delta\tau = \Delta\tau'$.

Although his deductions of the stress-stretch relations (2) and (3) from the respective laws of superposition are correct, Hencky does not provide explicit computations for either one. A proof for a generalized version of this deduction from the law of superposition was later given by **H. Richter** [20], who did extensive work on the matrix logarithm in finite elasticity [18, 19, 21, 22].

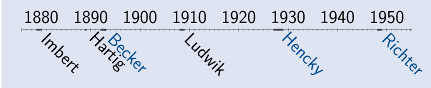
3 The prior work of G. F. Becker

However, the first known introduction of the logarithmic strain tensor to fully three-dimensional nonlinear elasticity is actually due to the famous geologist George Ferdinand Becker. In his 1893 article "The Finite Elastic Stress-Strain Function" [1, 16], he proposed a linear relation between the (material) logarithmic strain tensor $\log U$ and the *Biot stress tensor* T^{Biot} of the form

$$T^{\mathrm{Biot}}(U) = 2\,\mu \cdot \mathrm{dev}_3 \log(U) + \kappa \cdot \mathrm{tr}[\log U] \cdot \mathbb{1}\,. \qquad (5)$$

Becker used a systematic approach remarkably similar to Hencky's: by postulating a law of superposition for the stress-strain relation, a logarithmic constitutive law is deduced. Although it was reviewed in *Beiblätter zu Wiedemanns Annalen der Physik* [11] and cited in Lueger's *Lexikon der gesamten Technik* [13], Becker's work seems to have gone completely unnoticed in continuum mechanics until its recent rediscovery [16].

Note also that Becker's development of a detailed connection between stresses and the logarithm of the principal stretches in 1893 predates the publication of Ludwik's aforementioned derivation of the uniaxial logarithmic strain. The introduction of the logarithmic strain to elasticity theory is therefore currently misattributed in the literature. This error of attribution seems to originate from Hencky himself who, in a 1931 article [8, p. 175], refers to a brief section on plastic deformations in "Hütte: Des Ingenieurs Taschenbuch" [9] where Ludwik is cited. The same misattribution to Ludwik is given by Truesdell [23], who does not mention Becker at all.



Imbert	1880	**phenomenological** logarithmic model for the uniaxial **tension** of rubber
Hartig	1893	application of Imbert's law to **tension and compression** of rubber
Becker	1893	logarithmic **law of ideal elasticity** with respect to the **Biot stress**
Ludwik	1909	a short remark mentioning logarithmic strains
Hencky	1928	logarithmic **law of ideal elasticity** with respect to the **Cauchy stress**
Hencky	1929	logarithmic **law of ideal elasticity** with respect to the **Kirchhoff stress**, introduction of the **quadratic Hencky strain energy**
Richter	1949	characterization of the logarithmic strain tensor by the properties of its **deviatoric part**, further considerations of the **matrix logarithm**

4 References

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