10 Filters

Solution 10.1 Types of Filters

a) Frequency responses:

- top left: band-stop
  $\omega \to 0$, $\omega \to \frac{\omega_0}{2}$ (Nyquist-Frequency) : $|G(i\omega)| = 0$ dB $= 1 \Rightarrow$ pass band
  $0 < \omega < \frac{\omega_0}{2}$: $|G(i\omega)| = -80$ dB $\approx 0.0001 \Rightarrow$ stop band

- top right: low-pass
  $\omega \to 0$: $|G(i\omega)| = 0$ dB $= 1 \Rightarrow$ pass band
  $\omega \to \frac{\omega_0}{2}$: $|G(i\omega)| \to -\infty$ dB $= 0 \Rightarrow$ stop band

- bottom left: high pass
  $\omega \to 0$: $|G(i\omega)| \to -\infty$ dB $= 0 \Rightarrow$ stop band
  $\omega \to \frac{\omega_0}{2}$: $|G(i\omega)| \to 0$ dB $= 1 \Rightarrow$ pass band

- bottom right: band-pass
  $\omega \to 0$, $\omega \to \frac{\omega_0}{2}$: $|G(i\omega)| \to -\infty$ dB $= 0 \Rightarrow$ stop band
  $0 < \omega < \frac{\omega_0}{2}$: $|G(i\omega)| \approx 1$ dB $= 1 \Rightarrow$ pass band

Sampling frequency:

Amplitude responses end at $\omega \approx 30\frac{\text{rad}}{\text{s}}$. This is the Nyquist-Frequency. The Nyquist-Frequency equals

$$\omega_{Ny} = \frac{1}{2}\omega_0 \Rightarrow 60\frac{\text{rad}}{\text{s}} = \omega_0$$
$$\omega_0 = 2\pi f_0 \Rightarrow f_0 = \frac{\omega_0}{2\pi} = \frac{30}{\pi} Hz \approx 9,55 Hz$$

b)
- top left: high-pass
  ramp very low frequency

- top right: low-pass
  only the high frequency is suppressed

- bottom left: band-pass
  very low frequencies and high frequencies are suppressed

- bottom right:band-stop
  low and high frequencies are not suppressed

Solution 10.2 Properties of an Ideal Filter

a) stop-band: 0/ $-\infty$ dB
pass-band: 1/ 0 dB

b) Phase shifts are not desireable $\rightarrow 0^\circ$

c) Steepness: $-\infty$ (no transition range, just one frequency where the amplitude response suddenly changes!)

Solution 10.3 Requirements for Filters

a) In contrast to FIR-Filters, IIR-Filters can become unstable!

b) Filter with linear phase:
'System with pure dead time' or 'group propagation delay': $\tau_g = -\frac{d\varphi}{d\omega}$
Can exactly only be achieved through FIR-Filters!
Filter with linear phase means that the phase is shifted by a function of the following type:

$$\varphi(\omega) = \alpha \cdot \omega \qquad \text{(linear equation!)} \tag{1}$$

Mathematically a phase shift looks as follows:

$$y = A \cdot sin(\omega t + \underbrace{\varphi}_{\text{phase shift}}) \tag{2}$$

(1) in (2):

$$\begin{aligned} y &= A \cdot sin(\omega t + \alpha\omega) \\ &= A \cdot sin(\omega(t + \alpha)) \end{aligned}$$

As can be seen from this equation all frequencies are shifted in the same manner!
→ very important in acoustic environments because our ears are very sensitive to frequency-dependent phase differences!

c) An acausal filter needs values from the "future"to caluclate the current system/ filter output.
→ This is only possible for offline applications or systems where buffers can be used!
Advantage: The phase shift of a filter can be eliminated! (see script page 316)

Solution 10.4 Properties of IIR- and FIR-Filters

a) IIR

b) FIR

c) FIR

d) IIR

e) FIR

Solution 10.5 Impulse Response and Step Response of IIR- and FIR-Filters

a) Filter order: 3 $\Rightarrow u(k-3)$ most delayed signal
$\Rightarrow$ 4 terms (from 0...3) $\Rightarrow$ all terms have the same coefficient $\frac{1}{4}$ (averaging!)

$$y(k) = \frac{1}{4}u(k) + \frac{1}{4}u(k-1) + \frac{1}{4}u(k-2) + \frac{1}{4}u(k-3)$$

Impulse Response:

$$u(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned}
\rightarrow y(0) &= \frac{1}{4} & &= \frac{1}{4}u(0) + \underbrace{\frac{1}{4}u(-1)}_{=0} + \underbrace{\frac{1}{4}u(-2)}_{=0} + \underbrace{\frac{1}{4}u(-3)}_{=0} \\
y(1) &= \frac{1}{4} & &= + \underbrace{\frac{1}{4}u(1)}_{=0} + \frac{1}{4}u(0) + \underbrace{\frac{1}{4}u(-1)}_{=0} + \underbrace{\frac{1}{4}u(-2)}_{=0} \\
y(2) &= \frac{1}{4} \\
y(3) &= \frac{1}{4} \\
y(4) &= 0 \\
&\vdots \\
y(k>3) &= 0
\end{aligned}$$

$\Rightarrow$ FIR; after 3 time steps the impulse response reaches exactly 0 and keeps beeing zero!
Step- Response:

$$u(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned}
y(0) &= \frac{1}{4} \\
y(1) &= \frac{2}{4} = \frac{1}{2} \\
y(2) &= \frac{3}{4} \\
y(3) &= \frac{4}{4} = 1 \\
y(k>3) &= 1
\end{aligned}$$

$\rightarrow$ In every future step all delayed inputs maintain their value of 1!

b) New difference equation:

$$y(k) = 0.5u(k) + 0.25u(k-1) + 0.25u(k-2) + 0.25u(k-3)$$

Problem: The gain changes!
Step- Response:

$$u(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} y(0) &= 0,5 \\ y(1) &= 0,75 \\ y(2) &= 1 \\ y(3) &= 1,25 \\ y(k \geq 3) &= 1,25 \neq 1 \qquad \text{(value before the change of the first coefficient)} \end{aligned}$$

Solution: Decrease of other coefficients such that the sum of all coefficients keeps the same!
For example: 2 last coefficients reduced to $\frac{1}{8}$

$$\rightarrow y(k) = 0,5u(k) + \frac{1}{4}u(k-1) + \frac{1}{8}u(k-2) + \frac{1}{8}u(k-3)$$

Step-response: $y(0) = 0,5; y(1) = 0,75; y(2) = 0,875; \underline{\underline{y(k \geq 3) = 1}}$

c) $y(k) = 0,5u(k) + 0,5y(k-1)$ with $y(k < 0) = 0$

Impulse- Response:

$$u(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} y(0) &= 0,5u(0) + 0,5y(-1) &&= 0,5 \\ y(1) &= 0,5u(1) + 0,5y(0) &&= 0,5 \cdot 0,5 = 0,25 \\ y(2) &= 0 + 0,5y(1) &&= 0,5^3 \\ &\vdots \\ y(k) &= 0,5^{k+1} \end{aligned}$$

$y(k \rightarrow \infty) \rightarrow 0$; For any integer $k > 0$ $y(k) > 0 \Rightarrow$ IIR!

Step- Response:

$$u(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} y(0) &= 0,5u(0) + 0,5y(-1) = 0,5 \\ y(1) &= 0,5 + 0,5y(0) = \underbrace{0,5 + 0,5 \cdot 0,5}_{b_0 + b_0 a_1 = b_0(1+a_1)} = 0,75 \\ y(2) &= b_0 + \underbrace{b_0(1+a_1)}_{y(1)} a_1 = b_0(1 + a_1 + a_1^2) = 0,5(1 + 0,5 + 0,5^2) = 0,875 \end{aligned}$$

$$y(k) = b_0 \sum_{i=0}^{k} a_1^i = 0,5 \sum_{i=0}^{k} 0,5^i$$

$$\begin{aligned} \rightarrow s &= \sum_{i=0}^{k} 0,5^i \\ 0,5s &= \sum_{i=1}^{k+1} 0,5^i \\ s - 0,5s &= 0,5^0 - 0,5^{k+1} \\ \Rightarrow s = \frac{1-0,5^{k+1}}{1-0,5} &= 2\left(1-0,5^{k+1}\right) \end{aligned}$$

$y(k \rightarrow \infty) \rightarrow 1$; For every integer of k<∞, 1 is not exactly reached!

Approximated FIR-Filter: Impulse response values up to the desired order are used!

$y(0) = 0,5; y(1) = 0,25 = \frac{1}{4}; y(2) = 0,125 = \frac{1}{8}$

$$G_{FIR}^{(2)} = 0,5 + \frac{1}{4}z^{-1} + \frac{1}{8}z^{-2} = \frac{Y(z)}{U(z)}$$

$$y(k) = \frac{1}{2}u(k) + \frac{1}{4}u(k-1) + \frac{1}{8}u(k-2)$$

Step- Response:
$y(0) = \frac{1}{2}; y(1) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}; y(2) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$y(k \geq 2) = \frac{7}{8}$ (values maintain -> FIR)
Step response should at least tend to 1!
Solution: Increasing one or move of the coefficients such that

$$\sum_{i=0}^{2} b_i = 1$$

$$\text{i.e. } 0,5 + \frac{1}{4} + x = 1 \ ?$$

$$\Rightarrow x = \frac{1}{4} \text{ instead of } \frac{1}{8}$$

$$\Rightarrow G_{FIR}^{(2)} = 0,5 + \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2}$$

Solution 10.6 Acausal Filter

a) Which of the following 3 filters is causal, which is acausal and why?
To determine if a filter is causal or acausal, one way is to compare the maximum degree of the numerator and the denominator. If n is the maximum degree of the numerator and m the maximum degree of the denominator, a filter is causal, if the following statement holds:

$$m \geq n \ .$$

Now we transform the filter transfer functions into the writing with only positive exponents and determine, if they are causal or acausal:

$$G_1(z) = \frac{0.2z}{z - 0.8} \rightarrow \text{causal filter}$$

$$\begin{aligned} G_2(z) &= \frac{0.2z}{1 - 0.8z^{-1}} & \Big| \cdot \frac{z}{z} \\ &= \frac{0.2z^2}{z - 0.8} \rightarrow \text{acausal filter} \end{aligned}$$

$$\begin{aligned} G_3(z) &= \frac{0.2}{z^{-1} - 0.8z^{-2}} & \Big| \cdot \frac{z^2}{z^2} \\ &= \frac{0.2z^2}{z - 0.8} \rightarrow \text{acausal filter} \end{aligned}$$

b) Design an acausal filter of order 2, that has no phase shift. Sketch the response to a time-shifted step $\sigma(k-3)$ of the acausal filter and explain why there is no phase shift.
Acausal filter of order 2:

$$y(k) = \frac{1}{3}u(k-1) + \frac{1}{3}u(k) + \frac{1}{3}u(k+1) \ .$$

Now we calculate the step response to a time shifted unit step $\sigma(k-3) = \begin{cases} 1 & \text{for } k \geq 3 \\ 0 & \text{else} \end{cases}$:

$$\begin{aligned} k = 0 : y(0) &= \frac{1}{3}[u(-1) + u(0) + u(1)] \\ &= \frac{1}{3}[0 + 0 + 0] = 0 \\ k = 1 : y(1) &= \frac{1}{3}[u(0) + u(1) + u(2)] \\ &= \frac{1}{3}[0 + 0 + 0] = 0 \\ k = 2 : y(0) &= \frac{1}{3}[u(1) + u(2) + u(3)] \\ &= \frac{1}{3}[0 + 0 + 1] = \frac{1}{3} \\ k = 3 : y(3) &= \frac{1}{3}[u(2) + u(3) + u(4)] \\ &= \frac{1}{3}[0 + 1 + 1] = \frac{2}{3} \end{aligned}$$

$$k \geq 4 : y(4) = \frac{1}{3}[u(3) + u(4) + u(5)]$$
$$= \frac{1}{3}[1 + 1 + 1] = 1$$

Why is there no phase shift? To answer this question we take a closer look at the

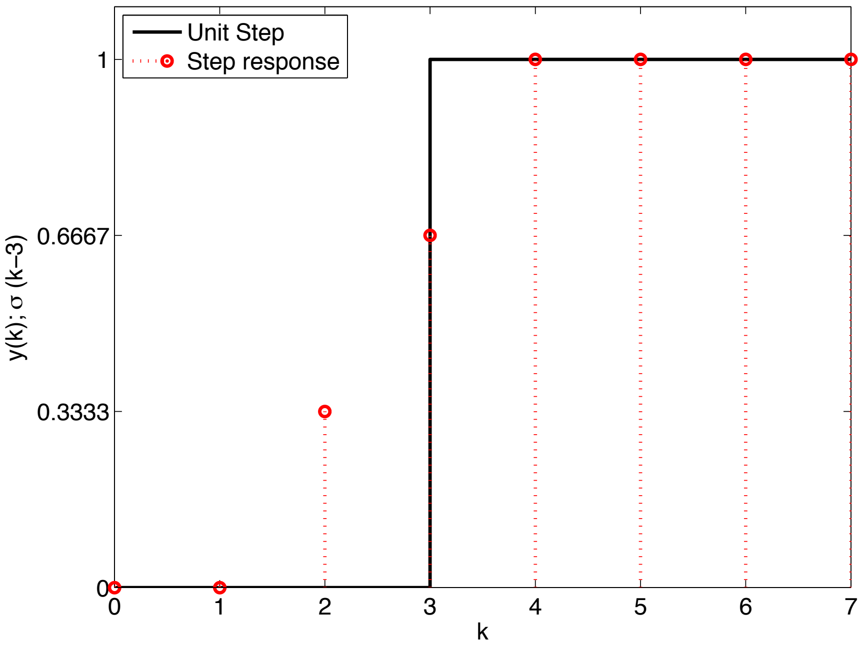

Figure 35: Step response of an acausal filter.

filter's transfer function:

$$G(z) = \frac{1}{3}\left(z^{-1} + 1 + z^{+1}\right) \qquad |z = e^{sT_0} \text{with} s = i\omega$$
$$= \frac{1}{3}\left(1 + e^{-i\omega T_0} + e^{i\omega T_0}\right) \qquad (1)$$

Now we use the following relationships:

$$e^{i\phi} = cos(\phi) + isin(\phi)$$
$$cos(-\phi) = cos(\phi)$$
$$sin(-\phi) = -sin(\phi) \ .$$

With this relationships equation 1 becomes:

$$G(z) = \frac{1}{3}(1 + cos(\omega T_0) - isin(\omega T_0) + cos(\omega T_0) + isin(\omega T_0))$$
$$= \frac{1}{3}(1 + 2cos(\omega T_0))$$

The equation to calculate the phase shift is:

$$\varphi = tan^{-1}\left(\frac{\text{Im}(G(z))}{\text{Re}(G(z))}\right) \ .$$

The imaginary part of our transfer function is exactly zero. So the phase shift becomes zero as well.

Solution 10.7 Bilinear Transformation

The equation to perform the bilinear transform is:

$$s = \frac{2}{T_0}\frac{1-z^{-1}}{1+z^{-1}} \ .$$

With the sampling time $T_0 = 1$ we can calculate the bilinear transform:

$$\begin{aligned}
G_{BT}(z) &= \frac{5}{1+5\cdot\left(\frac{2}{1}\frac{1-z^{-1}}{1+z^{-1}}\right)} && \text{|Transformation} \\
&= \frac{5(1+z^{-1})}{1+z^{-1}+10(1-z^{-1})} && \text{|Transformation} \\
&= \frac{5+5z^{-1}}{11-9z^{-1}} && \Big| \cdot \frac{\frac{1}{11}}{\frac{1}{11}} \\
&= \frac{\frac{5}{11}+\frac{5}{11}z^{-1}}{1-\frac{9}{11}z^{-1}}
\end{aligned}$$

We substitute the fractions, with their rounded values: $\frac{5}{11} \approx 0.45$ and $\frac{9}{11} \approx 0.82$:

$$G_{BT}(z) = \frac{0.45+0.45z^{-1}}{1-0.82z^{-1}} \ .$$

Now we compare $G_{BT}(z)$ with $G(z)$ with respect to the poles, zeros, gain and properness.

We start with the poles: Because both denominators are equal, there are no differences between the two discrete transfer functions regarding the stability. Both transfer functions have the same pole at $p_0 = 0.82$. Because $|p_0| < 1$, the transfer functions are stable.

Note: Through the bilinear transform no stability properties are affected, but the exact pole-location may differ (in general).

Comparison of the gain: Starting with the gain of $G(z)$:

$$y(k\to\infty) = \lim_{z\to 1}(z-1)\underbrace{\frac{0.9z^{-1}}{1-0.82z^{-1}}}_{G(z)}\underbrace{\frac{z}{z-1}}_{\text{Unit step}} = 5 \ .$$

Now the gain of the transfer function $G_{BT}(z)$ is calculated:

$$y(k\to\infty) = \lim_{z\to 1}(z-1)\underbrace{\frac{0.45+0.45z^{-1}}{1-0.82z^{-1}}}_{G_{BT}(z)}\underbrace{\frac{z}{z-1}}_{\text{Unit step}} = 5 \ .$$

$\rightarrow$ no difference - the correct gain of the continuous time system is reached.

Comparison of zeros:

$$\begin{aligned}
G(z) &= \frac{0.9}{z-0.82} \ \rightarrow \ \text{no zeros!} \\
G_{BT}(z) &= \frac{0.45z+0.45}{z-0.82} \ \rightarrow \ \text{one zero at } z=-1
\end{aligned}$$

The continuous time system has no zeros.

Properness:

$$G(z) = \frac{0.9}{z - 0.82} \rightarrow \text{ strictly proper}$$

Maximum degree of the numerator is smaller than the maximum degree of the denominator.

$$G_{BT}(z) = \frac{0.45z + 0.45}{z - 0.82} \rightarrow \text{ proper (not strictly! - system with feed-through)}$$

Maximum degree of the numerator is equal the maximum degree of the denominator.

Solution 10.8 Properties of Common Filters

a) Butterworth-Filter

b) Chebyshev
Type 1 (Ripples in the pass-band)
Type 2 (Ripples in the stop-band)

c) Bessel-Filter

d) Cauer-Filter

e) Bessel → Butterworth → Chebyshev → Cauer

Solution 10.9 Block-Diagram of a Time-Discrete Filter

a) Transform the transfer function into the form with only negative powers of z and evaluate the corresponding difference equation.

$$G(z) = \frac{Y(z)}{U(z)} = \frac{2z^3 + 3z^2}{z^3 + 2z^2 + z + 5} \qquad | \cdot \frac{z^{-3}}{z^{-3}}$$

$$\Leftrightarrow \frac{Y(z)}{U(z)} = \frac{2 + 3z^{-1}}{1 + 2z^{-1} + z^{-2} + 5z^{-3}} \qquad | \cdot \text{Transformation}$$

$$\Leftrightarrow Y(z) + 2Y(z)z^{-1} + Y(z)z^{-2} + 5Y(z)z^{-3} = 2U(z) + 3U(z)z^{-1}$$

This equation can easily be transformed into the discrete time domain:

$$y(k) + 2y(k-1) + y(k-2) + 5y(k-3) = 2u(k) + 3u(k-1) | \text{Transformation}$$

$$y(k) = 2u(k) + 3u(k-1) - (2y(k-1) + y(k-2) + 5y(k-3))$$

b) Sketch the corresponding block-diagram.

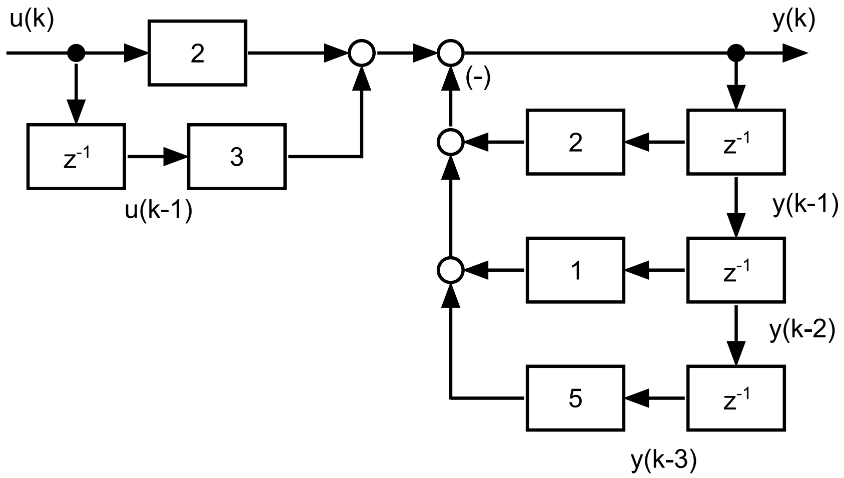

Figure 36: Block diagram of the filter $G(z)$.