Solution 9.5 Another DFT of a Periodic Sequence of Values

a) Fourier-Transform in matrix-vector notation.
$x(k)$ is a periodic sequence, that repeats after $N = 10$ values. For the discrete Fourier-Transform we can focus on only one period of the signal (in this case $k \in [0, 9]$).

$$x(k) = \begin{cases} 1 & 0 \leq k \leq 4 \\ 0 & 4 < k \leq 9 \end{cases}$$

The discrete Fourier-Transform is:

$$\mathrm{DFT}\{x(k)\} = X(n) = \sum_{k=0}^{9} x(k) W_{10}^{nk} \; .$$

Because all summands where $k > 4$ are zero, the equation can further be simplified:

$$\mathrm{DFT}\{x(k)\} = X(n) = \sum_{k=0}^{4} x(k) W_{10}^{nk} \; .$$

All values $x(k)$ equal one, such that:

$$\mathrm{DFT}\{x(k)\} = X(n) = \sum_{k=0}^{4} 1 \cdot W_{10}^{nk} \; ,$$

with $W_{10} = e^{i2\pi/10}$. The following equation system arises:

$$\begin{aligned}
X(0) &= W_{10}^{0} + W_{10}^{0} + W_{10}^{0} + W_{10}^{0} + W_{10}^{0} \\
X(1) &= W_{10}^{0} + W_{10}^{1} + W_{10}^{2} + W_{10}^{3} + W_{10}^{4} \\
X(2) &= W_{10}^{0} + W_{10}^{2\cdot 1} + W_{10}^{2\cdot 2} + W_{10}^{2\cdot 3} + W_{10}^{2\cdot 4} \\
X(3) &= W_{10}^{0} + W_{10}^{3\cdot 1} + W_{10}^{3\cdot 2} + W_{10}^{3\cdot 3} + W_{10}^{3\cdot 4} \\
X(4) &= W_{10}^{0} + W_{10}^{4\cdot 1} + W_{10}^{4\cdot 2} + W_{10}^{4\cdot 3} + W_{10}^{4\cdot 4} \\
X(5) &= W_{10}^{0} + W_{10}^{5\cdot 1} + W_{10}^{5\cdot 2} + W_{10}^{5\cdot 3} + W_{10}^{5\cdot 4} \\
X(6) &= W_{10}^{0} + W_{10}^{6\cdot 1} + W_{10}^{6\cdot 2} + W_{10}^{6\cdot 3} + W_{10}^{6\cdot 4} \\
X(7) &= W_{10}^{0} + W_{10}^{7\cdot 1} + W_{10}^{7\cdot 2} + W_{10}^{7\cdot 3} + W_{10}^{7\cdot 4} \\
X(8) &= W_{10}^{0} + W_{10}^{8\cdot 1} + W_{10}^{8\cdot 2} + W_{10}^{8\cdot 3} + W_{10}^{8\cdot 4} \\
X(9) &= W_{10}^{0} + W_{10}^{9\cdot 1} + W_{10}^{9\cdot 2} + W_{10}^{9\cdot 3} + W_{10}^{9\cdot 4} \; .
\end{aligned}$$

In matrix-vector notation, this equation system becomes:

$$\vec{X}(n) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W_{10}^{1} & W_{10}^{2} & W_{10}^{3} & W_{10}^{4} \\ 1 & W_{10}^{2} & W_{10}^{4} & W_{10}^{6} & W_{10}^{8} \\ 1 & W_{10}^{3} & W_{10}^{6} & W_{10}^{9} & W_{10}^{12} \\ 1 & W_{10}^{4} & W_{10}^{8} & W_{10}^{12} & W_{10}^{16} \\ 1 & W_{10}^{5} & W_{10}^{10} & W_{10}^{15} & W_{10}^{20} \\ 1 & W_{10}^{6} & W_{10}^{12} & W_{10}^{18} & W_{10}^{24} \\ 1 & W_{10}^{7} & W_{10}^{14} & W_{10}^{21} & W_{10}^{28} \\ 1 & W_{10}^{8} & W_{10}^{16} & W_{10}^{24} & W_{10}^{32} \\ 1 & W_{10}^{9} & W_{10}^{18} & W_{10}^{27} & W_{10}^{36} \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix} \; .$$

b) Simplification of the result.
For one discrete frequency n, the corresponding discrete Fourier-Transform value can be calculated as follows:

$$X(n) = \sum_{k=0}^{4} 1 \cdot (W_{10}^n)^k \ .$$

In more detail, the sum that has to be calculated looks like:

$$X(n) = (W_{10}^n)^0 + (W_{10}^n)^1 + (W_{10}^n)^2 + (W_{10}^n)^3 + (W_{10}^n)^4 \ . \tag{4}$$

If we multiply $X(n)$ with W_{10}^n we get:

$$W_{10}^n X(n) = (W_{10}^n)^1 + (W_{10}^n)^2 + (W_{10}^n)^3 + (W_{10}^n)^4 + (W_{10}^n)^5 \ . \tag{5}$$

Now we subtract equation 5 from 4:

$$\begin{aligned} X(n) - W_{10}^n X(n) &= (W_{10}^n)^0 - (W_{10}^n)^5 && |\text{Transformation} \\ X(n)[1 - W_{10}^n] &= 1 - (W_{10}^n)^5 && | : [1 - W_{10}^n] \\ X(n) &= \frac{1 - (W_{10}^n)^5}{1 - W_{10}^n} \ . \end{aligned}$$

Now we have to proof the following statement:

$$\frac{1 - (W_{10}^n)^5}{1 - W_{10}^n} = e^{-i4\pi\frac{n}{10}} \frac{sin(\frac{n}{2}\pi)}{sin(\frac{n}{10}\pi)} \ .$$

At first we use the Euler-Equation for the sine functions:

$$\begin{aligned} sin(\frac{n}{2}\pi) = sin(\frac{5n}{10}\pi) &= \frac{1}{2i}\left(e^{i\pi\frac{5n}{10}} - e^{-i\pi\frac{5n}{10}}\right) \ , \\ sin(\frac{n}{10}\pi) &= \frac{1}{2i}\left(e^{i\pi\frac{n}{10}} - e^{-i\pi\frac{n}{10}}\right) \ , \\ \frac{sin(\frac{n}{2}\pi)}{sin(\frac{n}{10}\pi)} &= \frac{e^{i\pi\frac{5n}{10}} - e^{-i\pi\frac{5n}{10}}}{e^{i\pi\frac{n}{10}} - e^{-i\pi\frac{n}{10}}} \ . \end{aligned}$$

Now we do some transformations:

$$\begin{aligned} \frac{1 - (W_{10}^n)^5}{1 - W_{10}^n} &= \frac{1 - e^{-i2\pi\frac{5n}{10}}}{1 - e^{-i2\pi\frac{n}{10}}} && | \cdot \frac{e^{i\pi\frac{n}{10}}}{e^{i\pi\frac{n}{10}}} \\ &= \frac{e^{i\pi\frac{n}{10}} - e^{-i\pi\frac{9n}{10}}}{e^{i\pi\frac{n}{10}} - e^{-i\pi\frac{n}{10}}} && |\text{Transformation} \\ &= e^{-i4\pi\frac{n}{10}} \frac{e^{i\pi\frac{5n}{10}} - e^{-i\pi\frac{5n}{10}}}{e^{i\pi\frac{n}{10}} - e^{-i\pi\frac{n}{10}}} \end{aligned}$$

Through some transformations, we see, that the expressions are equivalent.

Solution 9.6 Leakage Effect and Picket Fence Effect

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Solution 9.7 Leakage Effect

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