Crosspoint modification for multi-patch isogeometric analysis

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Introduction

Crosspoint modifications for general C^n continuous mortar coupling conditions arise in multi-patch geometries. This modification is constructed in such a way, that we decouple the Lagrange multipliers at the crosspoint to avoid a global coupling condition across all interfaces. In contrast to standard mortar techniques where the Lagrange multiplier space is often defined as a restricted trace space of the primal variable at the interface between the patches, the definition of a suitable Lagrange multiplier space for higher continuity also involves basis functions having a zero trace at the interface. Moreover, we recast the underlying B-Splines such that they preserve the higher order best approximation property across the interface and the crosspoint.

Extended mortar method

Concerning a general multi-patch situation with non-conformingly discretized subdomains $i \in \{1, \dots, N\}$, the C^n coupling conditions is given by

$$\mathcal{L}(u_{\mathrm{h}}^{(i)} - u_{\mathrm{h}}^{(j)}) = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}_{\mathrm{h}}^{(i,j)} \quad \text{with} \quad i \neq j,$$

where $\partial\mathcal{B}_{\mathrm{h}}^{(i,j)} = \partial\mathcal{B}_{\mathrm{h}}^{(i)} \cap \partial\mathcal{B}_{\mathrm{h}}^{(j)}$ denotes the common interface of two subdomains and the linear operator $\mathcal{L}$ represents the respective coupling conditions. The parametric B-spline coordinates $\bar{\boldsymbol{\xi}}^{(j)}$ are determined by a projection defined for a given $\bar{\boldsymbol{\xi}}^{(i)}$ as minimization problem

$$\|\bar{\boldsymbol{X}}_{\mathrm{h}}^{(i)}(\boldsymbol{\xi}^{(i)}) - \bar{\boldsymbol{X}}_{\mathrm{h}}^{(j)}(\boldsymbol{\xi}^{(j)})\| = \min_{\boldsymbol{\xi} \in \partial\tilde{\mathcal{B}}_{\mathrm{h}}^{(j)}} \|\bar{\boldsymbol{X}}_{\mathrm{h}}^{(i)}(\boldsymbol{\xi}^{(i)}) - \bar{\boldsymbol{X}}_{\mathrm{h}}^{(j)}(\boldsymbol{\xi})\|,$$

where $\bar{\boldsymbol{X}}_{\mathrm{h}}^{(i)}$ and $\bar{\boldsymbol{X}}_{\mathrm{h}}^{(j)}$ represent the isoparametric geometry representation of the interface. A classical mortar approach for the implementation of C^0 continuity, the mortar constraint for node $A = 1, \dots, \bar{\mathfrak{n}}^{(i)}$ at the interface $\partial\mathcal{B}_{\mathrm{h}}^{(i,j)}$ reads as

$$\Phi_0^{A,(i,j)} := h_A^{2(1-m)} \int_{\partial\mathcal{B}_{\mathrm{h}}^{(i,j)}} \left(R^{A,(i)}(\bar{\boldsymbol{\xi}}^{(i)}) R^{B,(i)}(\bar{\boldsymbol{\xi}}^{(i)}) u_B^{(i)} - R^{A,(i)}(\bar{\boldsymbol{\xi}}^{(i)}) R^{C,(j)}(\bar{\boldsymbol{\xi}}^{(j)}) u_C^{(j)} \right) \mathrm{d}\Gamma,$$

with the mesh size parameter h_A. The extension of the mortar constraint to preserve C^1 continuity can be written as

$$\Phi_1^{A,(i,j)} := \Phi_0^{A,(i,j)} + h_A^{2(2-m)} \sum_{k=1}^{d} \int_{\partial\mathcal{B}_{\mathrm{h}}^{(i,j)}} \left(R_{,k}^{A,(i)}(\bar{\boldsymbol{\xi}}^{(i)}) R_{,k}^{B,(i)}(\bar{\boldsymbol{\xi}}^{(i)}) u_B^{(i)} - R_{,k}^{A,(i)}(\bar{\boldsymbol{\xi}}^{(i)}) R_{,k}^{C,(j)}(\bar{\boldsymbol{\xi}}^{(j)}) u_C^{(j)} \right) \mathrm{d}\Gamma.$$

To guarantee C^2 continuity weakly, we require for each node A

$$\Phi_2^{A,(i,j)} := \Phi_1^{A,(i,j)} + h_A^{2(3-m)} \sum_{\substack{k,l=1 \\ k \geq l}}^{d} \int_{\partial\mathcal{B}_{\mathrm{h}}^{(i,j)}} \left(R_{,kl}^{A,(i)}(\bar{\boldsymbol{\xi}}^{(i)}) R_{,kl}^{B,(i)}(\bar{\boldsymbol{\xi}}^{(i)}) u_B^{(i)} - R_{,kl}^{A,(i)}(\bar{\boldsymbol{\xi}}^{(i)}) R_{,kl}^{C,(j)}(\bar{\boldsymbol{\xi}}^{(j)}) u_C^{(j)} \right) \mathrm{d}\Gamma.$$

Crosspoint modification

From a best approximation point of view we want that our reduced Lagrange multiplier space can still reproduce polynomials of order $p - 1$.

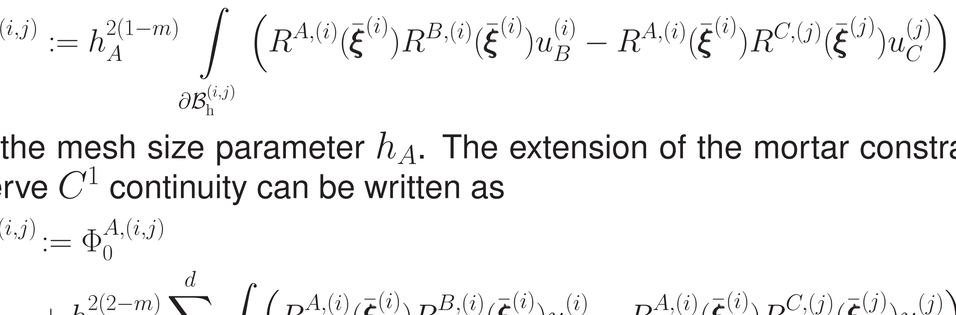

Figure: Evaluation of quadratic B-spline basis functions. and derivation in normal direction. Modified functions and derivatives at the crosspoint are colored in red. The dashed curves denote derivatives associated with the interior of the slave patch.

In case of a weak C^{l-1}, $1 \leq l \leq p$ coupling this means that we have to remove the first l basis functions on the interface but also functions associated with the interior of the subdomain. Now we want to modify the following next p such that the new reduced basis functions are given as

$$R_i^m = \sum_{j=1}^{l} c_{ij} R_j + R_{i+l}, \quad i = 1, \dots, p, \quad R_i^m = R_{i+l}, \qquad i > p$$

with coefficients matrix $\boldsymbol{C} \in \mathbb{R}^{p \times l}$, $[\boldsymbol{C}]_{ij} = c_{ij}$.

Phase separation in binary alloys

The Cahn-Hilliard equation describes the microstructural evolution of an alloy consisting of two species a and b. Therefore, a parameter $c : \mathcal{B} \times \mathcal{I} \to \mathbb{R}$ is introduced as $c := c_b = 1 - c_a$, where c_a and c_b are the mole fractions of the respective species. The evolution of c can be described by the balance equation

$$\dot{c} = \nabla \cdot (M \, \nabla\mu), \quad \forall (\boldsymbol{X}, t) \in \mathcal{B} \times \mathcal{I},$$

where μ denotes the effective chemical potential of b. The variational form follows in the usual straight-forward manner.

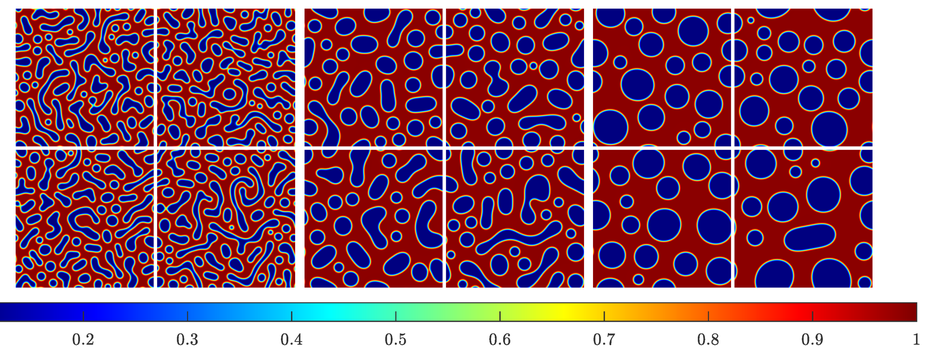

Figure: Solution of multi-patch simulation at different times $t = [10^3, 10^4, 10^5]\,\mathrm{s}$.

Grain growth in crystalline materials

The crystal growth equation is defined in terms of an order-parameter $\psi : \mathcal{B} \times \mathcal{I} \to \mathbb{R}$ which describes a local deviation from a reference mass density. Therefore, we introduce a Swift-Hohenberg energy function as

$$F(\psi) = \int_{\mathcal{B}} \frac{1}{4}\psi^4 + \frac{1}{2}(r+1)\psi^2 + \psi\Delta\psi + \frac{1}{2}\psi\Delta\Delta\psi \, \mathrm{d}V,$$

where the parameter r represents an undercooling of the system. To be specific, the crystal model is derived as a Wasserstein gradient flow of the Swift-Hohenberg energy

$$\dot{\psi} = \nabla \cdot \left(\psi^+ \nabla \frac{\delta F}{\delta \psi} \right), \quad \forall (\boldsymbol{X}, t) \in \mathcal{B} \times \mathcal{I},$$

The variational form follows in the usual straight-forward manner.

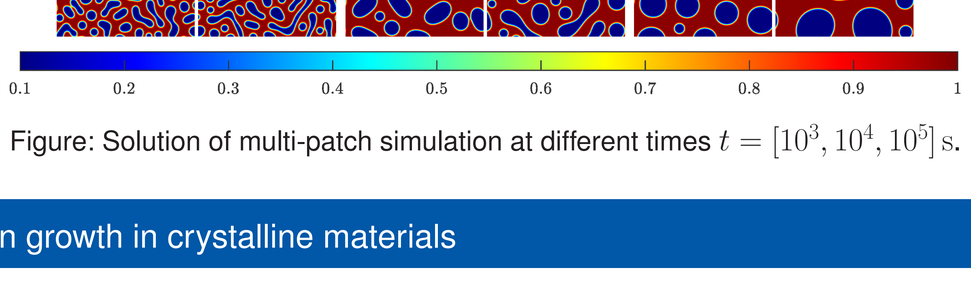

Figure: Solution of multi-patch simulation at different times $t = [50, 65, 100]$.

References:

[1] M. Dittmann, S. Schuß, B. Wohlmuth and C. Hesch. *Crosspoint modification for multi-patch isogeometric analysis*. Computer Methods in Applied Mechanics and Engineering, 360:112768, 2020.

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