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# Application of the discrete null space method to domain decomposition and large deformation contact problems

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We aim at the development of energy consistent 'mechanical' integrators for the differential-algebraic equations (DAEs) associated with the contact problems under consideration. First steps towards the energy consistent time integration of frictionless dynamic contact problems can be found in the works by Laursen & Chawla [4], Armero & Petöcz [1] and Laursen & Love [5]. Energy consistent mechanical integrators have been recently developed for DAE-formulations of constrained mechanical systems (e.g. Gonzalez [3]). These works are based on the direct discretization of the underlying DAEs leading to a saddle point system to be solved in each iteration of the iterative solution procedure. Due to the presence of Lagrange multipliers this approach leads to a large number of unknowns and potential conditioning problems. To remedy these drawbacks the discrete null space method (Betsch [2]) has been recently developed. We apply the discrete null space method to both domain decomposition in the framework of nonlinear elastodynamics and large deformation contact problems.



## 1 Kinematics

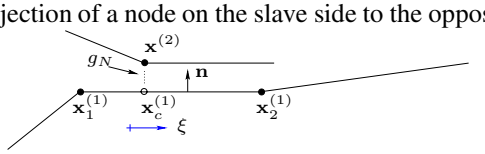
We deal with large deformation contact problems within the framework of nonlinear elastodynamics. Contact occurs if the deformation mappings of two opposing surfaces yield the same point in the current configuration. Accordingly, two material points $\mathbf{X}^1$ and $\mathbf{X}^2$ (a superscript denotes the respective surface) are in contact if $\boldsymbol{\varphi}^{(1)}(\mathbf{X}^{(1)},t) = \boldsymbol{\varphi}^{(2)}(\mathbf{X}^{(2)},t)$. The verification of a valid contact situation can be formulated using the Kuhn-Tucker conditions:

$$g_N \geq 0; \quad \lambda_N \geq 0; \quad g_N \lambda_N = 0 \tag{1}$$

Here, $g_N = (\boldsymbol{\varphi}^{(1)}(\mathbf{X}^{(1)},t) - \boldsymbol{\varphi}^{(2)}(\mathbf{X}^{(2)},t)) \cdot \mathbf{n}^1$ is the normal gap between the opposing surfaces and $\lambda_N$ characterizes the force acting between both bodies. Different approaches for the definition of the normal vector $\mathbf{n}^1$ are known either based on the underlying discretization or using some sort of smoothing of the surface, like Hermitian interpolation (see Wriggers [6]).

## 2 Formulation of the contact constraints

Two alternative approaches to the enforcement of the contact constraints in the semi-discrete system are considered next. In the NTS-method the constraint is defined by the orthogonal projection of a node on the slave side to the opposing master side. The associated nodal constraint $\Phi_A$ is defined by the normal gap function $g_N$. Several extentions of this formulation are known using segments of the contact surface and interpolating the gap-function over the segments.




The second formulation considered herein is the mortar method which relies on a segmentation of the discrete contact surface. In this connection linear transformations for the shape functions of the surface and the Lagrange multipliers are used within each segment to perform the integration over the contact surface. The corresponding constraints may be written in the form

$$\Phi_A = \bigcup_{\text{seg}} \sum_{B,C} \mathbf{n}_A \cdot [n_{AB}^{(1)} \boldsymbol{\varphi}_B^{(1)} - n_{AC}^{(2)} \boldsymbol{\varphi}_C^{(2)}] \tag{2}$$

where the mortar integrals are defined as

$$n_{AB}^{(1)} = \int_{\gamma_c^{seg}} N_A^{(1)}(\xi^{(1)}(\eta)) N_B^{(1)}(\xi^{(1)}(\eta)) \, d\gamma; \qquad n_{AC}^{(2)} = \int_{\gamma_c^{seg}} N_A^{(1)}(\xi^{(1)}(\eta)) N_C^{(2)}(\xi^{(2)}(\eta)) \, d\gamma \tag{3}$$

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## 3 Energy conservation

The equations of motion of the semi-discrete system under consideration may be formulated within the Hamiltonian framework. A mid-point type discretization of the corresponding DAEs is given by

$$\mathbf{q}_{n+1} = \mathbf{q}_n + h\mathbf{M}^{-1}\mathbf{p}_{n+\frac{1}{2}}; \quad \mathbf{p}_{n+1} = \mathbf{p}_n - h[\bar{\nabla}V(\mathbf{q}_n, \mathbf{q}_{n+1}) + \sum_A \lambda_A \bar{\nabla}\Phi_A(\mathbf{q}_n, \mathbf{q}_{n+1})]; \quad \mathbf{0} = \mathbf{\Phi}(\mathbf{q}_{n+1}) \tag{4}$$

where $(\bullet)_{n+\frac{1}{2}} = \frac{1}{2}\left((\bullet)_n + (\bullet)_{n+1}\right)$ and $\bar{\nabla}V$ is a discrete gradient of the potential function which accounts for hyperelastic material behavior and external forces. Similarly, $\bar{\nabla}\Phi_A$ denotes an appropriate discrete gradient of the contact constraints. The notion of a discrete gradient (or derivative) (cf. Gonzalez [1]) is essential for algorithmic energy conservation.

## 4 Discrete nullspace method

To remedy the drawback of a direct discretization of the DAEs at hand, a null-space method can be applied in order to eliminate the discrete Lagrange multipliers (cf. Betsch [2]). To this end a 'discrete null space matrix' $\mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1})$ needs be specified. By design,

$$\mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1})^T \bar{\nabla}\Phi_A(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathbf{0} \tag{5}$$

for all constraints. Thus the need to solve a saddle point system of the form

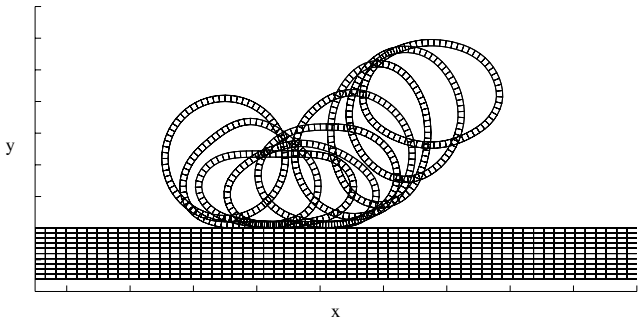
$$\begin{bmatrix} \mathbf{N}(\mathbf{q}_{n+1}, \boldsymbol{\lambda}) & h\mathbf{G}(\mathbf{q}_n, \mathbf{q}_{n+1})^T \\ \mathbf{G}(\mathbf{q}_{n+1}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{q}_{n+1} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} = - \begin{bmatrix} \mathbf{R}(\mathbf{q}_{n+1}) + h\mathbf{G}(\mathbf{q}_n, \mathbf{q}_{n+1})^T\boldsymbol{\lambda} \\ \mathbf{\Phi}(\mathbf{q}_{n+1}) \end{bmatrix} \tag{6}$$

in each iteration can be circumvented. Making use of the discrete null space matrix along with an appropriate reparametrization of the form $\mathbf{q}_{n+1} = \mathbf{F}_{q_n}(\mathbf{u})$ yields the reduced system

$$\mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1})^T \mathbf{N}(\mathbf{q}_{n+1}, \boldsymbol{\lambda}) D\mathbf{F}_{q_n}(\mathbf{u})\Delta\mathbf{u} = -\mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1})^T \mathbf{R}(\mathbf{q}_{n+1}) \tag{7}$$

## 5 Example

In the numerical example we consider the impact of an elastic ring on a stiff elastic foundation. A similar problem has previously been dealt with by Laursen & Chawla [4]. Concerning the material data we choose Young's modulus $E = 100$, Poisson's ratio $\nu = 0.1$ for the ring, and $E = 10^5$, $\nu = 0.1$ for the foundation. For both bodies geometrically nonlinear four node continuum elements are used. Application of the proposed time-stepping scheme in Section 3 results in a perfect conservation of energy.

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