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3D domain decomposition problems, consistent in time and space

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During the past years tremendous developments have been achieved in the field of domain decomposition problems as well as in the field of time integration schemes. In the present work actual developments in both fields are merged; the result of this is an extremely stable numerical method in space and in time.

1 Introduction

This section provides a short outline of the mechanical problem under consideration. For that reason, Figure 1 shows a schematic illustration of a typical body, subdivided into two parts, where both are tied together on the boundary Γ_d. It is assumed, that there exists for each part (i) a mapping $\boldsymbol{\varphi}^{(i)}(\boldsymbol{X}^{(i)}, t)$, connecting the current position at time t of a material point $\boldsymbol{X}^{(i)}$ with the reference configuration $\mathcal{B}^{(i)} \subset \mathbb{R}^3$. Furthermore, the boundaries $\Gamma^{(i)}$ can be subdivided into the Neumann boundaries $\Gamma_\sigma^{(i)}$, the Dirichlet boundaries $\Gamma_u^{(i)}$ and the previously mentioned domain decomposition boundary Γ_d. The

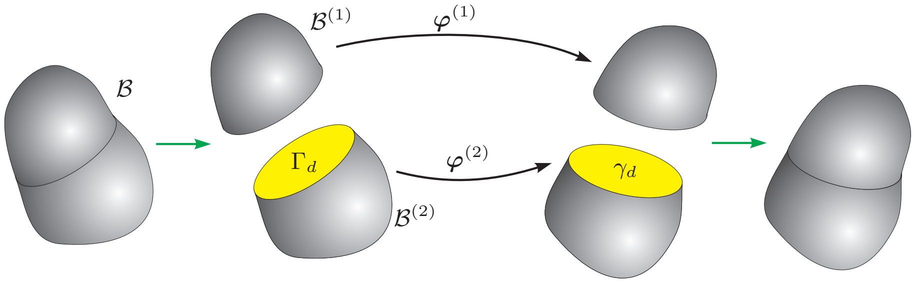

Fig. 1 Typical two-body problem

principle of virtual work for a general n-part problem is given by

$$\sum_{i=1}^{n} \left(G^{(i),\mathrm{dyn}} + G^{(i),\mathrm{int}} + G^{(i),\mathrm{ext}} + G^{(i),\mathrm{d}} \right) = 0 \tag{1}$$

obeying Hamilton's principle of stationary action in Lagrangian mechanics. The first term in equation (1) specifies the contribution of the inertia terms, the second term specifies the virtual work arising from the internal forces and the third term the virtual work of the external forces. Next, we focus our attention to the last term, which can be rewritten as follows

$$G^{\mathrm{d}} = \int_{\Gamma_{\mathrm{d}}^{(1)}} \boldsymbol{t}^{(1)} \cdot \left(\delta\boldsymbol{\varphi}^{(1)} - \delta\boldsymbol{\varphi}^{(2)} \right) \, \mathrm{d}\boldsymbol{\Gamma} \tag{2}$$

where the balance of linear momentum has been incorporated.

2 Spacial discretization

For the spacial discretization we consider standard finite elements in space (see Hughes [3]) for the displacements and their variations as well as for the first Piola-Kirchhoff tractions

$$\boldsymbol{\varphi}^{(i),\mathrm{h}} = \sum_{A \in \omega^{(i)}} N^A \left(\boldsymbol{X}^{(i)} \right) \boldsymbol{q}_A^{(i)}; \quad \delta\boldsymbol{\varphi}^{(i),\mathrm{h}} = \sum_{A \in \omega^{(i)}} N^A \left(\boldsymbol{X}^{(i)} \right) \delta\boldsymbol{q}_A^{(i)}; \quad \boldsymbol{t}^{(1),\mathrm{h}} = \sum_{A \in \bar{\omega}^{(1)}} N^A \left(\boldsymbol{X}^{(1)} \right) \boldsymbol{\lambda}_A \tag{3}$$

using polynomial, trilinear global shape functions $N^A(\boldsymbol{X}^{(i)}) : \mathcal{B} \to \mathbb{R}$ associated with nodes $A \in \omega^{(i)} = \{1, \ldots, n_{\mathrm{node}}^{(i)}\}$. Furthermore, $\bar{\omega}^{(i)} \subset \omega^{(i)}$ denotes the set of nodes on the respective internal interfaces and $\boldsymbol{\lambda}_A$ the associated nodal values of

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the first Piola-Kirchhoff tractions. Inserting the approximations (3) into (2) yields the contribution of the discrete interface forces to the virtual work:

$$G^{\mathrm{d}}(\boldsymbol{\varphi}^{\mathrm{h}}, \boldsymbol{t}^{(1),\mathrm{h}}; \delta\boldsymbol{\varphi}^{\mathrm{h}}) = \int\limits_{\Gamma_{\mathrm{d}}^{(1)}} \boldsymbol{t}^{(1),\mathrm{h}} \cdot \left(\delta\boldsymbol{\varphi}^{(1),\mathrm{h}} - \delta\boldsymbol{\varphi}^{(2),\mathrm{h}}\right) \mathrm{d}\boldsymbol{\Gamma} = \sum_{A} \boldsymbol{\lambda}_A \cdot \left(\sum_{B} n^{AB} \delta\boldsymbol{q}_B^{(1)} - \sum_{C} n^{AC} \delta\boldsymbol{q}_C^{(2)}\right) \tag{4}$$

where n^{AB} and n^{AC} denote the so-called mortar integrals given by

$$n^{AB} = \int\limits_{\Gamma_{\mathrm{d}}^{(1)}} N^A\left(\boldsymbol{X}^{(1)}\right) N^B\left(\boldsymbol{X}^{(1)}\right) \mathrm{d}\boldsymbol{\Gamma}; \quad n^{AC} = \int\limits_{\Gamma_{\mathrm{d}}^{(1)}} N^A\left(\boldsymbol{X}^{(1)}\right) N^C\left(\boldsymbol{X}^{(2)}\right) \mathrm{d}\boldsymbol{\Gamma} \tag{5}$$

3 Time discretization

Next, as special augmentation technique (see Hesch & Betsch [2]) has to be utilized to ensure algorithmic conservation of linear and angular momentum as well as energy. The equations of motion now reads

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = -\nabla V(\boldsymbol{q}) - \left(\mathrm{D}_1\boldsymbol{g}(\boldsymbol{q}, \boldsymbol{d})\right)^T \boldsymbol{\lambda}; \quad \boldsymbol{0} = \left(\mathrm{D}_2\boldsymbol{g}(\boldsymbol{q}, \boldsymbol{d})\right)^T \boldsymbol{\lambda}; \quad \boldsymbol{0} = \boldsymbol{g}(\boldsymbol{q}, \boldsymbol{d}) \tag{6}$$

Here, $\boldsymbol{g}$ denote the combined constraints for the mortar method and the augmented coordinates $\boldsymbol{d}$. The full discrete system can now be easily applied by using a mid-point type rule (see Betsch & Hesch [1]).

4 Example

A pressurised quarter of a half-sphere with a Youngs modulus of $E = 10000$ and a Poissons ratio of $\nu = 0.4$ has been simulated. In Figure 4 a sequence of configurations is displayed. On the left side of the diagrams the time history of three

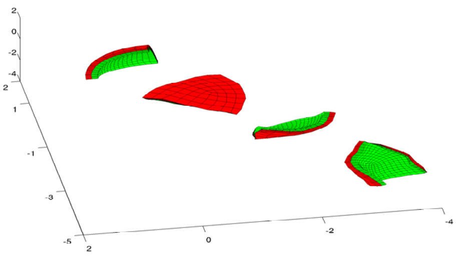
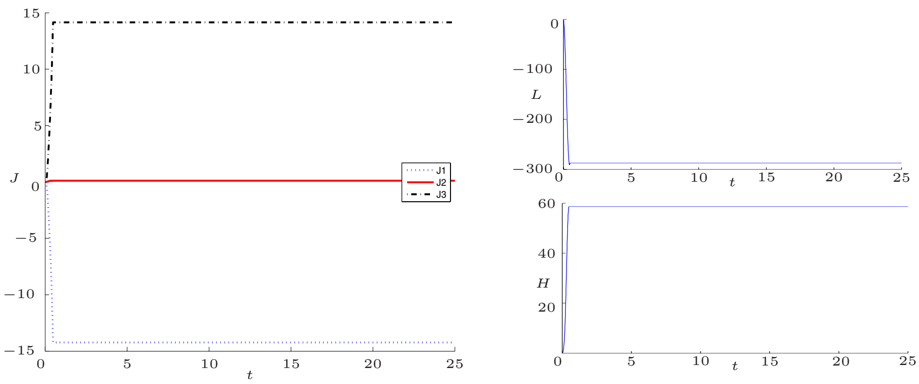


Fig. 2 Sequence of configurations



components of the angular momentum, on the right the history of linear momentum and total energy are displayed. All quantities are conserved up to the numerical roundoff errors.

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