# Multidimensional coupling: A variationally consistent approach to fiber-reinforced materials

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## Introduction

Fiber reinforced materials are subjected to various physical mechanisms on different scales, depending on the size, orientation and distribution within a suitable matrix material. Applications contain, e.g. steel reinforced ultra-high performance concrete or fiber reinforced polymers. Besides such technical materials, many biological tissues are reinforced by certain types of fibers. A model containing fibers fully resolved as a discretized continua, subsequently referred to as Cauchy continuum approach, is far out of the range of today's computational capabilities. Therefore, we develop a model containing the fibers as continuum degenerated 1D beams within the matrix material in the sense of an overlapping domain decomposition method.

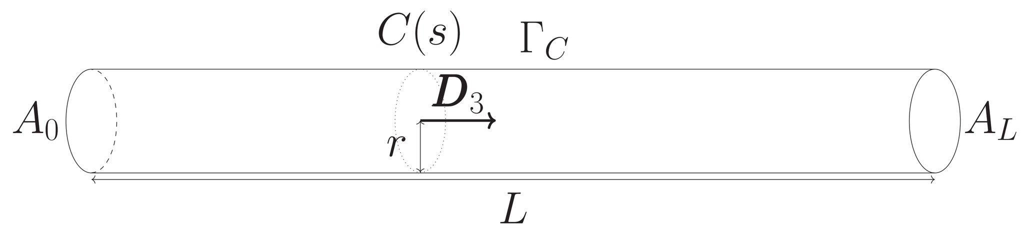
## Solid mechanics

The matrix material is described by the 3D non-linear continuum mechanics. Here, $\Omega_0$ is the reference configuration and $\Omega$ the current configuration with boundaries $\partial\Omega_0$ and $\partial\Omega$, respectively. The derivative of the mapping $\boldsymbol{\varphi}$ yields the deformation gradient $\boldsymbol{F} = \nabla\boldsymbol{\varphi}$. The strain energy function is given by $\Psi := \Psi(\boldsymbol{F}, \operatorname{cof}\boldsymbol{F}, \det\boldsymbol{F})$. The virtual work of the matrix material is:

$$\delta\Pi^{int} + \delta\Pi^{ext} = \int_{\Omega_0} \boldsymbol{P} : \nabla\delta\boldsymbol{\varphi}\,\mathrm{d}V - \int_{\Omega_0} \boldsymbol{B}_{ext}\cdot\delta\boldsymbol{\varphi}\,\mathrm{d}V - \int_{\Gamma^\sigma} \boldsymbol{T}_{ext}\cdot\delta\boldsymbol{\varphi}\,\mathrm{d}A\,.$$

## Continuum degenerated beam formulation

For the fiber material we degenerate the general continuum mechanical framework as introduced above to a beam formulation. As we intend to embed fibers with a length-to-diameter ratio of 20 as standard for e.g. fiber reinforced polymers, it is reasonable that we restrict the kinematics of the 3D continuum along the fiber direction to a beam-like kinematic. In particular, we use the theory of geometrically exact beams, also known as Cosserat beams.



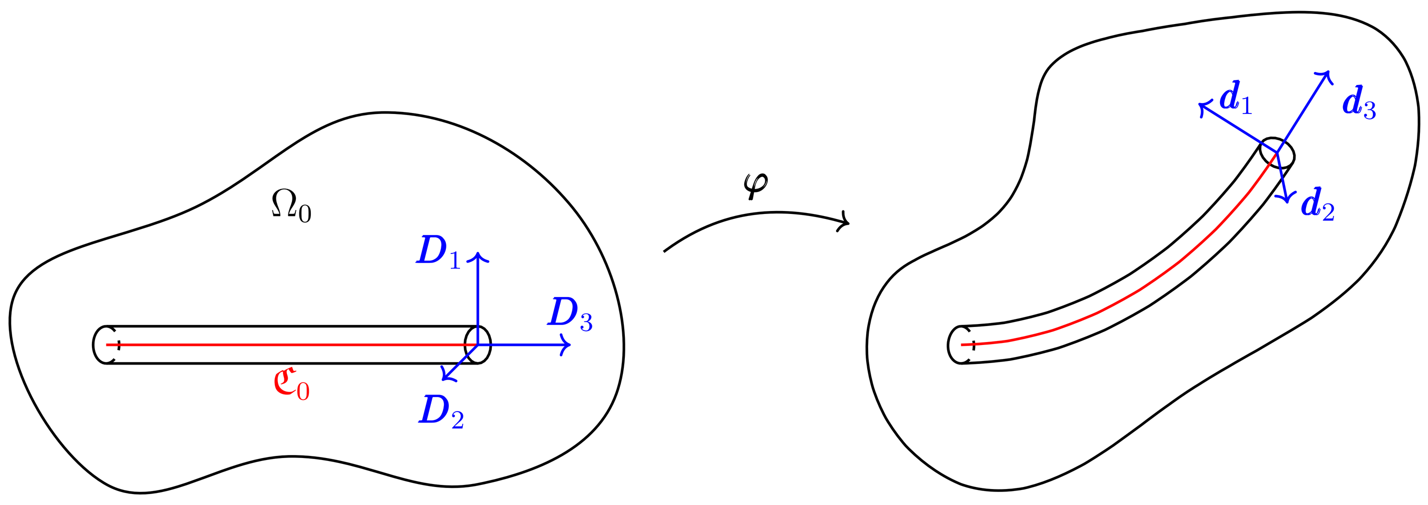

The reference and current configuration are $\tilde{\Omega}_0$ and $\tilde{\Omega}$, respectively. The motion of the beam is given by $\tilde{\boldsymbol{x}}(\theta^\alpha, s) = \tilde{\boldsymbol{\varphi}}(s) + \theta^\alpha\,\boldsymbol{d}_\alpha(s)$ and the deformation gradient reads $\tilde{\boldsymbol{F}} = \nabla\tilde{\boldsymbol{x}}$. The strain energy function is given by $\Psi := \Psi(\boldsymbol{\Gamma}, \boldsymbol{K})$ with the axial shear $\boldsymbol{\Gamma}$ and the torsional-bending strain $\boldsymbol{K}$. The virtual work of the fiber material is:

$$\delta\tilde{\Pi}^{int} = \int_{\mathfrak{C}_0} \left(\tilde{\boldsymbol{R}}\,\tilde{\boldsymbol{N}}\right)\cdot\delta\tilde{\boldsymbol{\varphi}}' - \left(\tilde{\boldsymbol{\varphi}}' \times \tilde{\boldsymbol{R}}\,\tilde{\boldsymbol{N}}\right)\cdot\delta\boldsymbol{\phi} + \left(\tilde{\boldsymbol{R}}\,\tilde{\boldsymbol{M}}\right)\cdot\delta\boldsymbol{\phi}'\,\mathrm{d}s,$$

$$\delta\tilde{\Pi}^{ext} = -\int_{\mathfrak{C}_0} \left(\bar{\tilde{\boldsymbol{n}}}\cdot\delta\tilde{\boldsymbol{\varphi}} + \bar{\tilde{\boldsymbol{m}}}\cdot\delta\boldsymbol{\phi}\right)\,\mathrm{d}s - [\![\boldsymbol{n}^e_{ext}\cdot\delta\tilde{\boldsymbol{\varphi}} + \boldsymbol{m}^e_{ext}\cdot\delta\boldsymbol{\phi}]\!]_0^L\,.$$

## Multidimensional coupling model

Starting from a surface-to-volume coupling formulation for the matrix/beam system, we derive a reduced surrogate model, where the new coupling constraints defined on the beam centerline involve the deformation gradient and transfer both the linear forces and moments of the beam to the matrix. The Lagrange multiplier $\boldsymbol{\mu}$, physically interpreted as the interface load, is defined on the beam mantle $\boldsymbol{\mu}(\theta, s) = \bar{\boldsymbol{\mu}}(s) + \Sigma(s)\,\boldsymbol{N}(\theta)$ with the mean




to the resulting force in the cross-section and, therefore, couples position of the beam center-line to the matrix, $\Sigma\,\boldsymbol{N}$ contains the coupled stresses, shears and hydrostatic pressure, and is thus responsible for transition of the beam bending and torsion to the matrix. The virtual work of the coupling constraints is:

$$\begin{aligned}\delta\Pi_C = &\int_{\mathfrak{C}_0} \left[\delta\bar{\boldsymbol{\mu}}\cdot(\boldsymbol{\varphi}_c - \tilde{\boldsymbol{\varphi}}) + \bar{\boldsymbol{\mu}}\cdot(\delta\boldsymbol{\varphi} - \delta\tilde{\boldsymbol{\varphi}})\right]|C|\,\mathrm{d}s\\ &+ \int_{\mathfrak{C}_0} \left[\boldsymbol{\Sigma} : \left(\delta\boldsymbol{F}_c - [\delta\boldsymbol{\phi}]_\times\tilde{\boldsymbol{R}}\right) + \delta\boldsymbol{\Sigma} : \left(\boldsymbol{F}_c - \tilde{\boldsymbol{R}}\right)\right]|A|\,\mathrm{d}s,\end{aligned}$$

$$\delta\Pi_A = [\![\delta\bar{\boldsymbol{\mu}}_e\cdot(\boldsymbol{\varphi}_e - \tilde{\boldsymbol{\varphi}})\,|A_e| + \bar{\boldsymbol{\mu}}_e\cdot(\delta\boldsymbol{\varphi}_e - \delta\tilde{\boldsymbol{\varphi}})\,|A_e|]\!]_0^L\,,$$

with circular means $\boldsymbol{\varphi}_c$, $\boldsymbol{F}_c$ and endface means $\bar{\boldsymbol{\mu}}_e$, $\bar{\boldsymbol{\mu}}_e$.

## Beam/matrix system

Finally, we add all virtual works for the coupled matrix/beam system:

$$\delta\Pi^{total}(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}}, \tilde{\boldsymbol{R}}, \mathfrak{n}, \mathfrak{m}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}}_e, \tilde{\boldsymbol{\mu}}_\tau, \tilde{\boldsymbol{\mu}}_n) = 0\,,$$

where a Hu-Washizu type method yields $\mathfrak{n}$ and $\mathfrak{m}$. This system is large, as we have to deal with the degrees of freedom of the matrix material (3 per node in the 3D continuum), and the 21 unknowns including 9 Lagrange multipliers per node along the beam center line. Thus, we aim at a two step static condensation procedure: first, we condense the beam equations and eliminate the corresponding Lagrange multipliers in the continuous system. The remaining equations for the constraints and the constitutive laws for the beam are condensed in the discrete setting, such that finally only the matrix degrees of freedom remain and the beam is fully condensed and yields a second gradient model.

## Torsion test

- Matrix - Mooney-Rivlin material, beam - Saint-Venant-Kirchhoff material
- Geometrical data via second moment of inertia of the 1D Beam is visible
- External moment on beam applied

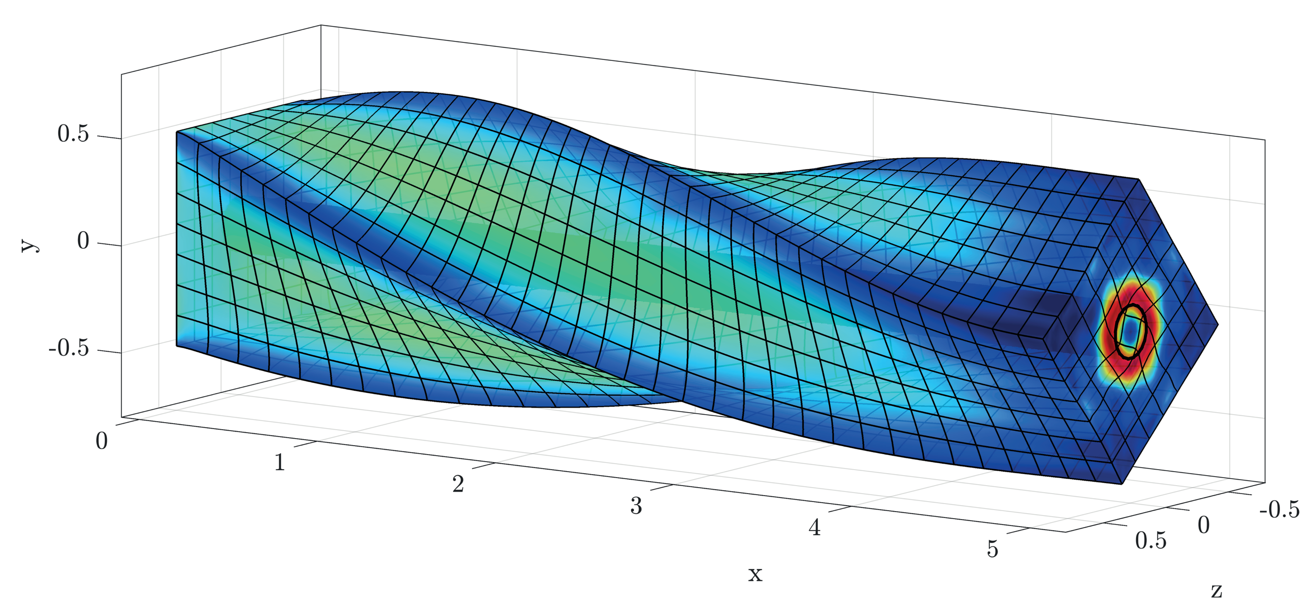



Figure: **Torsion Test:** Von Mises stress distribution.

**References:**

[1] U. Khristenko, S. Schuß, M. Krüger, F. Schmidt, B. Wohlmuth and C. Hesch. *Multidimensional coupling: A variationally consistent approach to fiber- reinforced materials.* Computer Methods in Applied Mechanics and Engineering, 382 (2021).