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Non-linear space-time elasticity

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Introduction

Classical time stepping schemes for dynamic simulations are highly inefficient on modern cluster architectures and thus, waste of energy. Here, we introduce a novel space-time formulations, able to calculate large deformations and displacements with high efficiency using structured and unstructured meshes. Therefore, we introduce tesseract elements for 4-dimensional calculations, removing one of the major bottlenecks in parallel computations. Moreover, stability of time-stepping schemes depend on the accumulation of local approximation errors in each time-step, in contrast to space-time formulation, characterized by enhanced stability and robustness.

General formulation

Starting point is always Hamilton's law of varying action

$$\delta\mathcal{L} = 0,$$

where $\mathcal{L}$ the Lagrangian defined as action integral of the kinetic minus the strain energy. This general formula includes in general all kind of multibody systems, i.e. rigid bodies, beams, shells and continua. Integration by parts in space and time provide all necessary information on the boundaries, including initial and end points in temporal direction. Assuming that the kinetic energy $T(\boldsymbol{v})$ is convex and differentiable, we can introduce the conjugate function $T^*(\boldsymbol{\pi})$, where $\boldsymbol{\pi}$ refers to the linear momentum. The Legendre transformed Lagrangian reads now

$$\mathcal{L}^*(\boldsymbol{\varphi},\boldsymbol{\pi}) = \int_{\mathfrak{B}_0} \boldsymbol{\pi}\cdot\nabla_t(\boldsymbol{\varphi}) - \frac{1}{2}\rho_0^{-1}\|\boldsymbol{\pi}\|^2 - \Psi(\boldsymbol{F})\,\mathrm{d}W - A^{\mathrm{ext}},$$

where $\boldsymbol{\varphi}$ denotes the deformation map, ρ_0 the material density, $\Psi(\boldsymbol{F})$ the strain energy function in terms of the deformation gradient $\boldsymbol{F}$ and A^{ext} all external contributions.

Stabilization and discretization

Continuous hyperbolic systems have a unique solution $\boldsymbol{\varphi} \in \mathcal{H}^{1,1}_{0;0,}(\mathfrak{B}_0)$, if

$$\mathcal{H}^{1,1}_{0;0,} = L^2([0,T];\mathcal{H}^1_0(\Omega_0)) \cap \mathcal{H}^1_{0,}([0,T];L^2(\Omega_0)),$$

where $\mathcal{H}^1_{0,}([0,T];L^2(\Omega_0))$ refers to zero initial conditions, and if the space of admissible or trial functions $\delta u \in \mathcal{H}^{1,1}_{0;,0}(\mathfrak{B}_0)$ with

$$\mathcal{H}^{1,1}_{0;,0} = L^2([0,T];\mathcal{H}^1_0(\Omega_0)) \cap \mathcal{H}^1_{,0}([0,T];L^2(\Omega_0)),$$

where $\mathcal{H}^1_{,0}([0,T];L^2(\Omega_0))$ denotes zero terminal conditions. This has already been shown in the fundamental work of Ladyzhenskaya [2]. To fulfill this condition, a stabilization using a time upwind formulation in the discrete setting

$$\boldsymbol{\varphi}^{\mathrm{h}}_{\mathrm{st}}(\boldsymbol{X},t) = \sum_{s=1}^{\mathfrak{n}(n+1)} \varphi^{\mathrm{st}}_s \boldsymbol{R}^s_{\mathrm{st}}(\boldsymbol{X},t), \quad \boldsymbol{\pi}^{\mathrm{h}}_{\mathrm{st}}(\boldsymbol{X},t) = \sum_{s=1}^{\mathfrak{n}(n+1)} \pi^{\mathrm{st}}_s \boldsymbol{R}^s_{\mathrm{st}}(\boldsymbol{X},t),$$

is required, using variations $\delta\boldsymbol{\varphi}^{\mathrm{h}} + \theta h\nabla_t(\delta\boldsymbol{\varphi}^{\mathrm{h}})$ and $\delta\boldsymbol{\pi}^{\mathrm{h}} + \theta h\nabla_t(\delta\boldsymbol{\pi}^{\mathrm{h}})$, where θ is a positive constant. Introducing the functionals A^{ij}_{h}, $i,j = 1,2$, given by

$$A^{11}_{\mathrm{h}}(\boldsymbol{\varphi}^{\mathrm{h}},\delta\boldsymbol{\varphi}^{\mathrm{h}}) = \int_{\mathfrak{B}_0} \boldsymbol{P}^{\mathrm{h}} : \nabla_X(\delta\boldsymbol{\varphi}^{\mathrm{h}} + \theta h\nabla_t(\delta\boldsymbol{\varphi}^{\mathrm{h}}))\,\mathrm{d}W,$$

$$A^{12}_{\mathrm{h}}(\boldsymbol{\pi}^{\mathrm{h}},\delta\boldsymbol{\varphi}^{\mathrm{h}}) = \int_{\mathfrak{B}_0} \nabla_t(\boldsymbol{\pi}^{\mathrm{h}}) \cdot (\delta\boldsymbol{\varphi}^{\mathrm{h}} + \theta h\nabla_t(\delta\boldsymbol{\varphi}^{\mathrm{h}}))\,\mathrm{d}W,$$

$$A^{21}_{\mathrm{h}}(\boldsymbol{\varphi}^{\mathrm{h}},\delta\boldsymbol{\pi}^{\mathrm{h}}) = \int_{\mathfrak{B}_0} \nabla_t(\boldsymbol{\varphi}^{\mathrm{h}}) \cdot (\delta\boldsymbol{\pi}^{\mathrm{h}} + \theta h\nabla_t(\delta\boldsymbol{\pi}^{\mathrm{h}}))\,\mathrm{d}W,$$

$$A^{22}_{\mathrm{h}}(\boldsymbol{\pi}^{\mathrm{h}},\delta\boldsymbol{\pi}^{\mathrm{h}}) = -\int_{\mathfrak{B}_0} \rho_0^{-1}\boldsymbol{\pi}^{\mathrm{h}} \cdot (\delta\boldsymbol{\pi}^{\mathrm{h}} + \theta h\nabla_t(\delta\boldsymbol{\pi}^{\mathrm{h}}))\,\mathrm{d}W$$

and the linear form l_{h} as discrete external contributions, the discrete form of the space-time problem reads

$$A_{\mathrm{h}}(\boldsymbol{\varphi}^{\mathrm{h}},\boldsymbol{\pi}^{\mathrm{h}},\delta\boldsymbol{\varphi}^{\mathrm{h}},\delta\boldsymbol{\pi}^{\mathrm{h}}) = \begin{pmatrix} A^{11}_{\mathrm{h}}(\boldsymbol{\varphi}^{\mathrm{h}},\delta\boldsymbol{\varphi}^{\mathrm{h}}) + A^{12}_{\mathrm{h}}(\boldsymbol{\pi}^{\mathrm{h}},\delta\boldsymbol{\varphi}^{\mathrm{h}}) \\ A^{21}_{\mathrm{h}}(\boldsymbol{\varphi}^{\mathrm{h}},\delta\boldsymbol{\pi}^{\mathrm{h}}) + A^{22}_{\mathrm{h}}(\boldsymbol{\pi}^{\mathrm{h}},\delta\boldsymbol{\pi}^{\mathrm{h}}) \end{pmatrix} = \begin{pmatrix} l_{\mathrm{h}}(\delta\boldsymbol{u}^{\mathrm{h}}) \\ 0 \end{pmatrix}.$$

Modified Newton iteration

Introducing the residual vector $\boldsymbol{\mathcal{R}}(\boldsymbol{z})$, where $\boldsymbol{z} = (\tilde{\boldsymbol{\varphi}}^{\mathrm{T}}, \tilde{\boldsymbol{\pi}}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{\mathrm{n}_\varphi+\mathrm{n}_\pi}$ contains the degrees of freedom, we finally obtain the corresponding algebraic problem which reads as follows: find $\boldsymbol{z} \in \mathbb{R}^{\mathrm{n}_\varphi+\mathrm{n}_\pi}$ such that

$$\boldsymbol{\mathcal{R}}(\boldsymbol{z}) = \boldsymbol{0}.$$

Among the wide range of algorithms for the solution of root-finding problems, Newton's method remains unquestionable one of the most powerful tools. However, it requires an initial guess z_0 sufficiently close to the searched root to ensure convergence. Especially for space-time systems, this information is in general not available in advance. To overcome this problem, modified (or damped) Newton iterations of the form

$$\boldsymbol{z}_{k+1} = \boldsymbol{z}_k + \lambda_k\boldsymbol{d}_k, \quad k = 0,1,\ldots,$$

are applied, where λ_k is a suitable step-size and $\boldsymbol{d}_k = -(\partial_{\boldsymbol{z}}\boldsymbol{\mathcal{R}}(\boldsymbol{z}_k))^{-1}\boldsymbol{\mathcal{R}}(\boldsymbol{z}_k)$ the Newton direction.

Tesseract elements: Torus example

In order to exactly represent the torus, we use for the parametrization and the approximation NURBS shape-functions of order $\boldsymbol{q} = (2,1,2,2)$ such that it is resolved by 34 elements along the torus azimuth, 18 elements along the perimeter of the tube, 2 elements along the thickness direction and 10 elements along the time direction, obtaining a resolution of the space-time cylinder from a total of 12240 tesseracts.

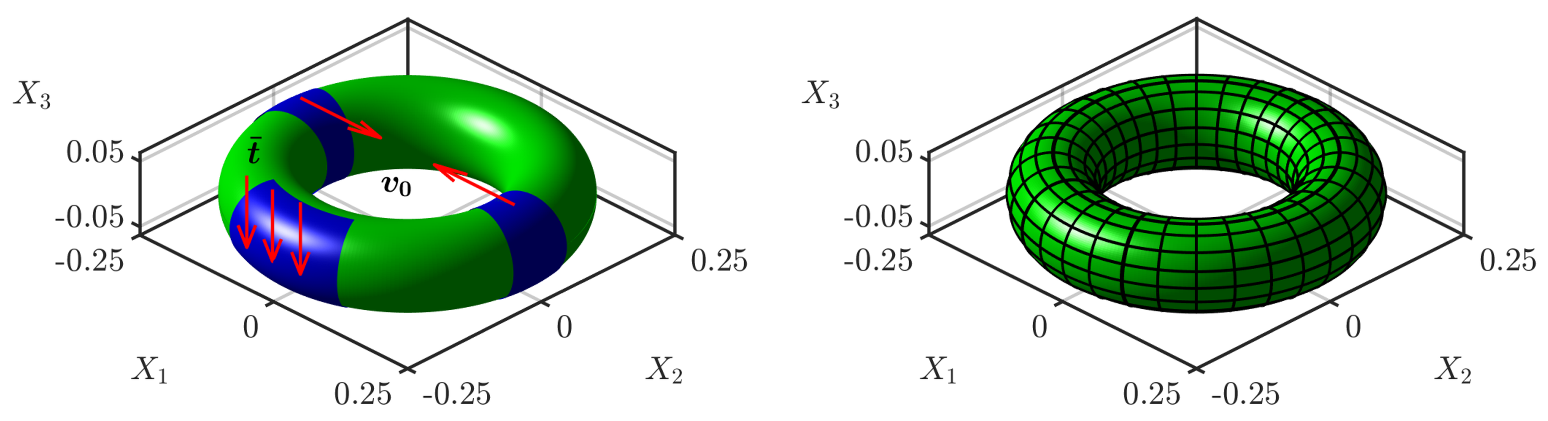

Figure: Three dimensional configuration (left) and spatial computational mesh (right).

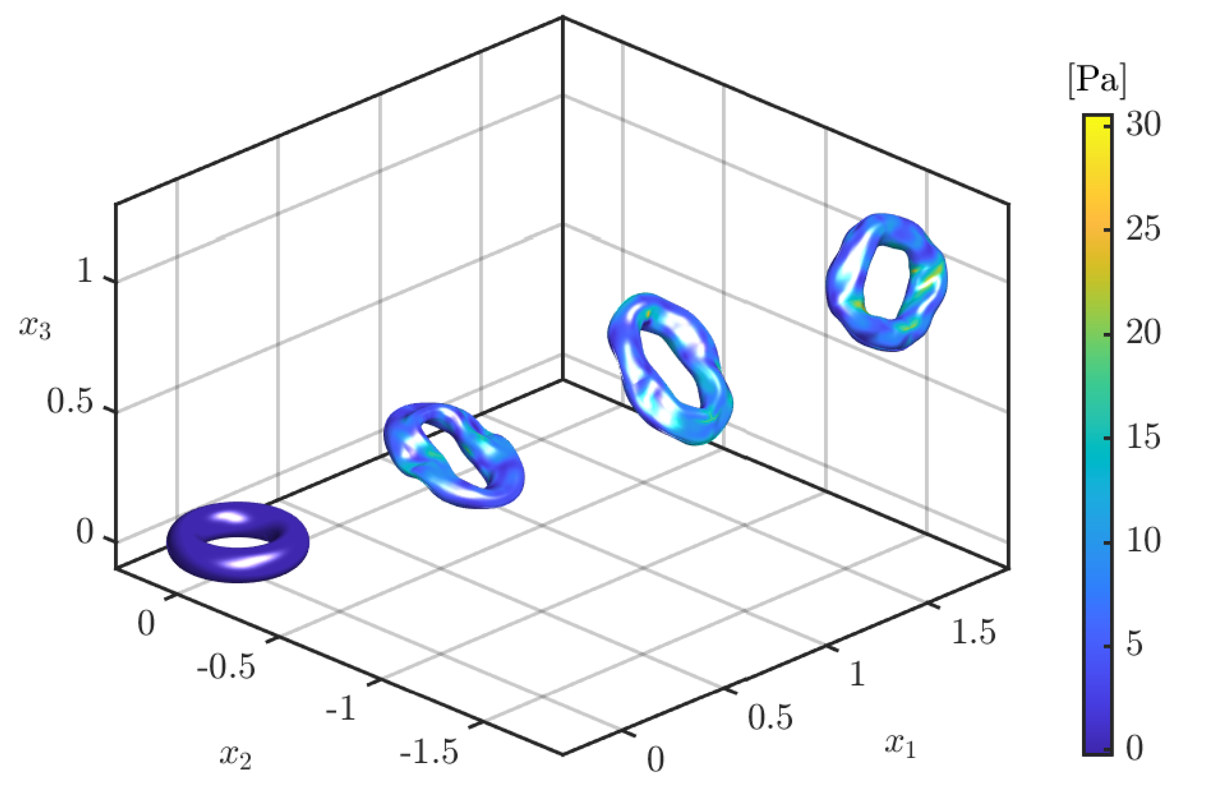

Figure: Von Mises stress distribution, configurations at time $t = 0, 1/3, 2/3, 1$ (left to right).

The modified Newton algorithm used to solve the non-linear system, achieved within 13 iteration steps the termination criterion $\|\boldsymbol{\mathcal{R}}(\boldsymbol{z}_k)\| \leq 10^{-10}$.

References:

[1] S. Schuß, S. Glas and C. Hesch. *Non-linear space-time elasticity.* International Journal for Numerical Methods in Engineering, DOI:10.1002/nme.7194, 2023.

[2] O.A. Ladyzhenskaya. *The boundary Value Problems of Mathematical Physics.* Springer, 1985.