# Isogeometric analysis for large deformation contact problems

**M. Dittmann, C. Hesch and P. Betsch**

## Introduction

Isotropic Hyperelastic materials

- Strain energy function to describe nonlinear mechanical behaviour
- Same response of the material in all directions
- Mooney Rivlin material as a special case of the Ogden material

Discrete contact formulation

- NTS-discretization as five-node contact element
- Active set strategy to fulfill the Kuhn-Tucker-Karush condition
- Lagrange multipliers to add contact contributions to the weak form

Isogeometric analysis

- NURBS based shape functions
- KTS-method to enforce contact
- Definiton of an exact torus geometry

## Constitutive law

Mooney-Rivlin material [1]

- Strain energy function

$$\Psi(I_1, I_2, J) = c_1(I_1 - 3) + c_2(I_2 - 3) - b\ln(J) + c\,(J-1)^2$$

- Growth conditions

$$\Psi(J \to +\infty) \to +\infty \qquad \text{and} \qquad \Psi(J \to 0^+) \to +\infty$$

- Second Piola-Kirchhoff stress tensor and elasticity tensor

$$\boldsymbol{S} = 2\frac{\partial \Psi}{\partial \boldsymbol{C}} \qquad \mathbb{C} = 4\frac{\partial^2 \Psi}{\partial \boldsymbol{C}^2}$$

## Contact formulation

Constraints

- Non-penetration condition

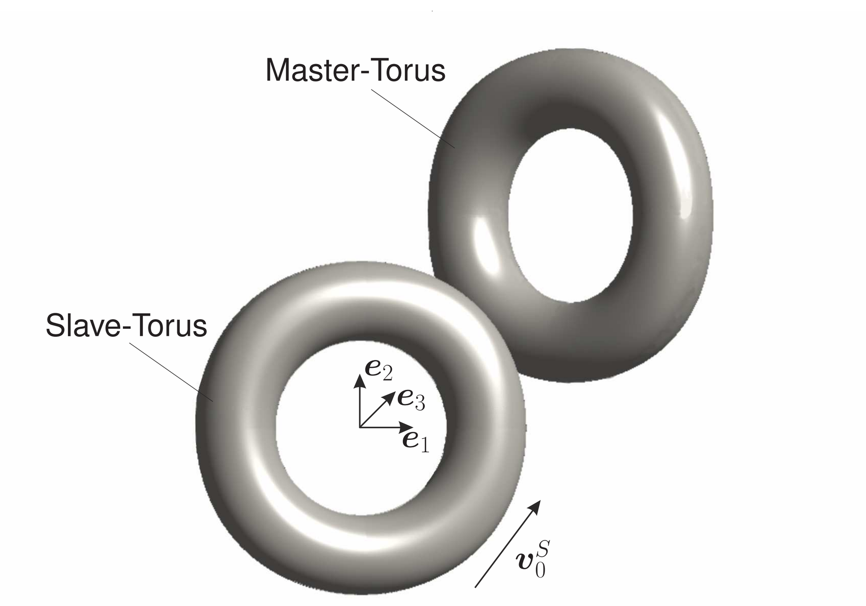
$$g_N = \left(\boldsymbol{x}^S - \bar{\boldsymbol{x}}^M\right)\cdot \bar{\boldsymbol{n}}^M \geq 0$$

- Kuhn-Tucker-Karush condition

$$g_N \geq 0 \quad p_N \leq 0 \quad g_N\, p_N = 0$$

- Lagrange multiplier method

$$\Pi_c = \int_{\Gamma_c^{akt}} p_N\, g_N\, \mathrm{d}\Gamma$$




NTS-discretization

- Interpolation of Master-surface

$$\boldsymbol{x}_h^M(\xi_1, \xi_2, t) = \sum_{i=1}^{4} \hat{N}(\xi_1, \xi_2)\, \boldsymbol{x}_l^M(t)$$

- Five-node-contact element

$$\boldsymbol{x}_{(s)} = \left[\boldsymbol{x}^S\ \boldsymbol{x}_1^M\ \boldsymbol{x}_2^M\ \boldsymbol{x}_3^M\ \boldsymbol{x}_4^M\right]_{(s)}^{\mathrm{T}}$$

- Weak form of contact contribution

$$C_c = \sum_{s=1}^{n_c} p_{N(s)}\, \delta g_{N(s)} A_{(s)} = \sum_{s=1}^{n_c} \boldsymbol{\eta}_{(s)}^{\mathrm{T}} \boldsymbol{C}_{(s)}$$

- Discrete contact constraints

$$\boldsymbol{\Phi} = \left[g_{N(1)}, \ldots, g_{N(s)}, \ldots, g_{N(n_C)}\right]^{\mathrm{T}} = \boldsymbol{0}$$

- Weak form

$$G_h(\boldsymbol{x}, \boldsymbol{\eta}) = \boldsymbol{\eta}^{\mathrm{T}}\left[\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{R}(\boldsymbol{x}) + \boldsymbol{C}(\boldsymbol{x})\right] = 0$$

## Discretization in time

Mid-point type rule

$$\boldsymbol{x}_{n+1} - \boldsymbol{x}_n = \Delta t \boldsymbol{v}_{n+\frac{1}{2}}$$
$$\boldsymbol{M}(\boldsymbol{v}_{n+1} - \boldsymbol{v}_n) = -\Delta t\left[\boldsymbol{R}(\boldsymbol{x}_{n+1}, \boldsymbol{x}_n) + \boldsymbol{C}(\boldsymbol{x}_{n+1}, \boldsymbol{x}_n)\right]$$
$$\boldsymbol{0} = \boldsymbol{\Phi}(\boldsymbol{x}_{n+1})$$

## Isogeometric analysis

NURBS [2]

- Shape functions

$$R_{i,j,k}^{p,q,r} = \frac{N_{i,p}(\xi)\, M_{j,q}(\eta)\, L_{k,r}(\zeta)\, w_{i,j,k}}{\sum_{\hat{i}=1}^{n} N_{\hat{i},p}(\xi) \sum_{\hat{j}=1}^{m} N_{\hat{j},q}(\eta) \sum_{\hat{k}=1}^{l} N_{\hat{k},r}(\zeta)\, w_{\hat{i},\hat{j},\hat{k}}}$$

- B-Splines basis functions are recursively defined as

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)$$

- Beginning with

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if} \quad \xi_i \leq \xi \leq \xi_{i+1}\,, \\ 0 & \text{otherwise} \end{cases}$$

- Knot vectors

$$\Xi = [\xi_1, \ldots, \xi_{n+p+1}] \ \ \mathcal{H} = [\eta_1, \ldots, \eta_{m+q+1}] \ \ \mathcal{Z} = [\zeta_1, \ldots, \zeta_{l+r+1}]$$

- Control points

$$\tilde{\boldsymbol{B}} = \left[\tilde{\boldsymbol{B}}_{1,1,1}, \ldots, \tilde{\boldsymbol{B}}_{i,j,k}, \ldots, \tilde{\boldsymbol{B}}_{n,m,l}\right]^{\mathrm{T}}$$

- NURBS Solid

$$\boldsymbol{\mathcal{B}}(\xi, \eta, \zeta) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} R_{i,j,k}^{p,q,r}(\xi, \eta, \zeta)\, \tilde{\boldsymbol{B}}_{i,j,k}$$

## Numerical results

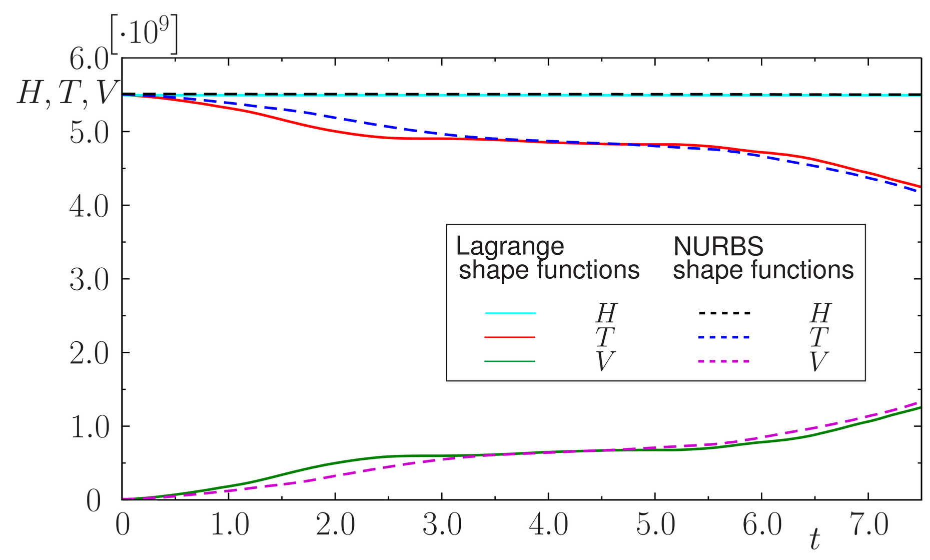
Impact of two hollow tori

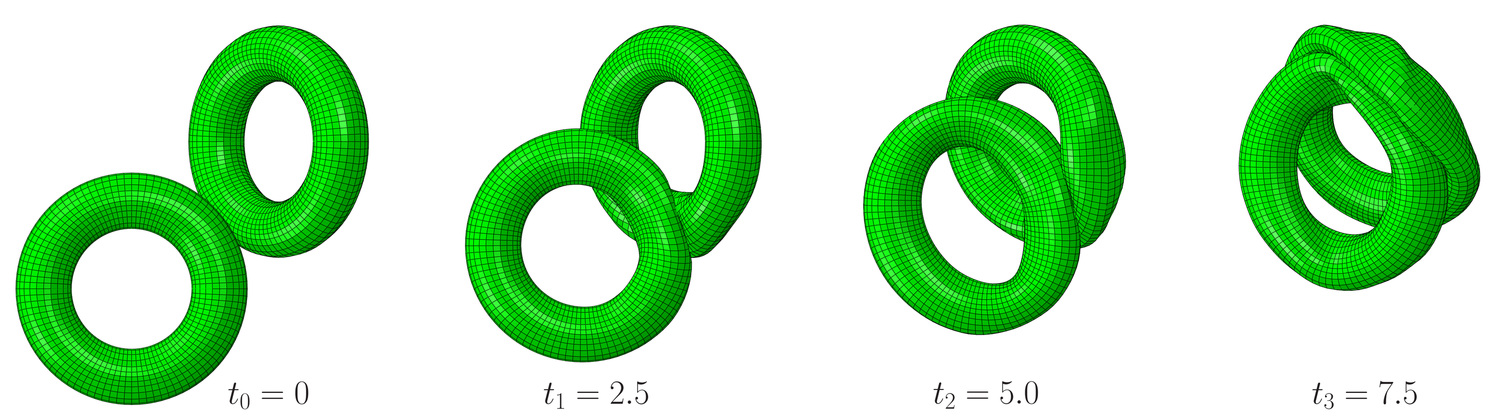
- Material parameter

$$E = 1.0507 \cdot 10^6$$
$$\nu = 0.4562$$
$$\rho_0 = 60$$

- Initial velocity

$$\boldsymbol{v}_0^S = [20\ 15\ 0]^{\mathrm{T}}$$




### References

G. A. Holzapfel
Nonlinear Solid Mechanics.
*Willey, Chichester*, 2000

T.J.R. Hughes, J.A. Cottrell, Y. Bazilevs
Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement.
*Comput. Methods Appl. Mech. Engrg.*, 194:4135-4195, 2005

**University of Siegen**
**Chair of Computational Mechanics**

**Prof. Dr.-Ing. habil. C. Hesch**