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Variational Space-Time Elements for Large-Scale Systems

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Introduction

Space-time formulation [2]

- Continuum mechanical approach in $\mathbb{R}^{n+1}$
- Variational formulation in space and time
- Space-time adaptivity and multigrid solver possible

Application to a wide range of transient systems

- Thermal conduction
- Nonlinear elasticity
- Fluids and material science

Initial boundary value problem

- Reformulation as boundary value problem in the space-time domain
$$\tilde{\mathcal{B}} := \mathcal{B} \times I \subset \mathbb{R}^{n+1}$$
- Find $\boldsymbol{u} \colon \tilde{\mathcal{B}} \rightarrow \mathbb{R}^m$ such that
$$\boldsymbol{A}(\boldsymbol{u}(\boldsymbol{y}), \boldsymbol{y}) \nabla_t \boldsymbol{u}(\boldsymbol{y}) + \mathbf{L}\boldsymbol{u}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{y}) \quad \forall \boldsymbol{y} \in \tilde{\mathcal{B}}$$
- supplemented by
$$\boldsymbol{u}(\boldsymbol{y}) = \tilde{\boldsymbol{g}}(\boldsymbol{y}) \quad \text{on} \quad \boldsymbol{y} \in \partial\tilde{\mathcal{B}}^u$$
$$(\mathbf{R}\boldsymbol{u})(\boldsymbol{y}) = \boldsymbol{h}(\boldsymbol{y}) \quad \text{on} \quad \boldsymbol{y} \in \partial\tilde{\mathcal{B}}^\sigma$$

where $\boldsymbol{y} = (\boldsymbol{x}, t)$ represents an event in the space-time.

Thermal conduction

- Definition of solution and weighting functions
$$\mathcal{S} = \{\theta \in \mathsf{H}^1(\tilde{\mathcal{B}}) \,|\, (\theta = \bar{\theta} \text{ on } \partial\tilde{\mathcal{B}}^\theta) \wedge (\theta = \tilde{\theta} \text{ on } \partial\tilde{\mathcal{B}}^{t_1})\}$$
$$\mathcal{V} = \{\delta\theta \in \mathsf{H}^1(\tilde{\mathcal{B}}) \,|\, (\delta\theta = 0 \text{ on } \partial\tilde{\mathcal{B}}^\theta) \wedge (\delta\theta = 0 \text{ on } \partial\tilde{\mathcal{B}}^{t_1})\}$$
- Weak problem: Find $\theta \in \mathcal{S}$ such that
$$B(\theta, \delta\theta) = L(\delta\theta) \quad \forall\, \delta\theta \in \mathcal{V}$$

where
$$B(\theta, \delta\theta) := \int_{\tilde{\mathcal{B}}} \delta\theta \rho c_p(\theta) \nabla_t(\theta) \,\mathrm{d}W + \int_{\tilde{\mathcal{B}}} \nabla_{\boldsymbol{x}}(\delta\theta) \cdot \boldsymbol{K}(\theta) \nabla_{\boldsymbol{x}}(\theta) \,\mathrm{d}W$$
$$L(\delta\theta) := \int_{\tilde{\mathcal{B}}} \delta\theta R \,\mathrm{d}W + \int_{\partial\tilde{\mathcal{B}}^Q} \delta\theta \bar{Q} \,\mathrm{d}\Gamma$$

Here, $\mathrm{d}W = \mathrm{d}V\,\mathrm{d}t$ and $\mathrm{d}\Gamma = \mathrm{d}A\,\mathrm{d}t$.

Discretisation

- Finite dimensional approximation in space-time
$$\theta^{\mathrm{h}} = \sum_{A \in \omega} N^A(\boldsymbol{y}) \theta_A \quad \text{and} \quad \delta\theta^{\mathrm{h}} = \sum_{A \in \omega} N^A(\boldsymbol{y}) \delta\theta_A$$
where $A \in \omega = \{1, \ldots, n_{\mathrm{node}}\}$ are events in space-time.
- Discrete problem: Find $\theta^{\mathrm{h}} \in \mathcal{S}^h$ such that
$$B\left(\theta^{\mathrm{h}}, N^A\right) = L\left(N^A\right) \quad \forall\, A \in \omega$$

where
$$B\left(\theta^{\mathrm{h}}, N^A\right) = \int_{\tilde{\mathcal{B}}^{\mathrm{h}}} N^A \rho c_p(\theta^{\mathrm{h}}) \nabla_t(N^B) \,\mathrm{d}W \theta_B + \int_{\tilde{\mathcal{B}}^{\mathrm{h}}} \nabla_{\boldsymbol{x}}(N^A) \cdot \boldsymbol{K}(\theta^{\mathrm{h}}) \nabla_{\boldsymbol{x}}(N^B) \,\mathrm{d}W \theta_B$$
$$L\left(N^A\right) = \int_{\tilde{\mathcal{B}}^{\mathrm{h}}} N^A R \,\mathrm{d}W + \int_{\partial\tilde{\mathcal{B}}^{Q,\mathrm{h}}} N^A \bar{Q} \,\mathrm{d}\Gamma$$

is valid.

Multigrid approach

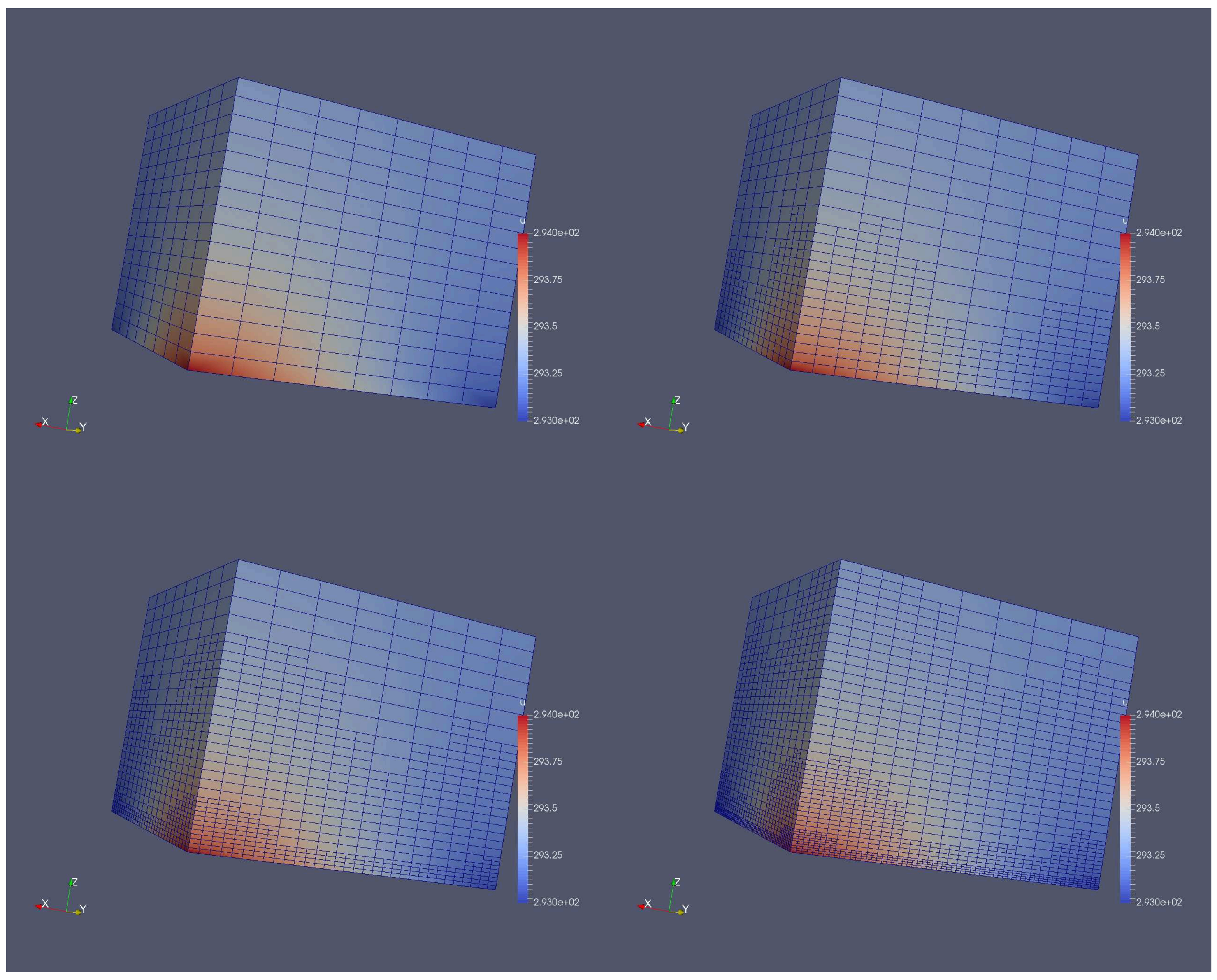
From a multigrid point of view, the main difficulty for space-time solution is that in time we have to deal with a purely convective problem (after possible reduction to a first order system). The multigrid approach considered here is therefore based on a stabilization in time. Here, we follow the approach of [1] and add diffusion in time.

For the application on continuous Lagrangian finite elements in time and space-time, we construct the coarser approximation spaces using semi-geometric multigrid methods. These methods create a nested hierarchy of finite element spaces based on a hierarchy of possibly non-nested meshes. To this end, a discrete (pseudo-)L^2-projection operator between the finite element spaces related to the non-nested meshes is computed. The weights of the resulting scaled mass-matrices are then used for defining the coarse level spaces based on linear combinations of the fine level basis functions.

Multigrid performance for thermal systems

Material parameters for copper at room temperature are used:

- Conductivity $\boldsymbol{K} = 400\boldsymbol{I}$
- Heat capacity $c_p = 385$
- Density $\rho = 8920$



Space-time adaptive approach based on a gradient error indicator.

References

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