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AN ENERGY CONSISTENT HYBRID SPACE-TIME FINITE ELEMENT METHOD FOR NONLINEAR THERMO-VISCOELASTODYNAMICS

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Abstract. We present a new space-time fe discretisation of a finite thermo-viscoelastic coupled sytem. The discretisation fulfills a nonlinear stability estimate with respect to a Lyapunov function called the total energy. The coupled sytem of differential equations first-order in time consists of the equations of motion, the entropy evolution equation and an evolution equation for a viscous internal variable. A new hyprid Galerkin (hG) approximation composed of a continuous Galerkin (cG) and a new discontinuous Galerkin (dG) method supplemented by particular time approximations in the constitutive laws leads to long-time stable numerical schemes.

1 INTRODUCTION

For performing a time discretisation an advantage of the fe method in time is that higher-order accurate time approximations can be obtained in a natural way. A further advantage is that a cG method in time is inherently energy-momentum consistent for the equations of motion (see Reference[1]). Hence the cG method is a natural starting point to construct higher-order and energy-momentum-conserving time integrators (see Reference[2]). These time integrators, derived from the enhanced Galerkin (eG) method, turned out to be well suited for computing long-time runs in nonlinear elastodynamics. An energy consistent time approximation is also of great advantage for simulating dissipative material (see References[3,4]). We show that an energy consistent discretisation by using a new enhanced hybrid Galerkin (ehG) method is also advantageous for nonlinear dynamical systems with dissipation arising from conduction of heat and from a memory effect of viscous material.

2 CONSTITUTIVE EQUATIONS

In a Lagrangian setting, we consider a thermo-viscoelastic material described by Fourier's law of heat conduction together with an internal variable formulation. We start with the relative internal energy $e(\eta_t, \mathbf{C}_t, \mathbf{\Gamma}_t) = [\Theta_t - \Theta_\infty]\,\eta_t + \Psi(\Theta_t, \mathbf{C}_t, \mathbf{\Gamma}_t)$ emanating from a Legendre transform of the free energy $\Psi(\Theta_t, \mathbf{C}_t, \mathbf{\Gamma}_t)$ with respect to the temperature Θ_t. The fields Θ_∞

and η_t denote the constant environment temperature field and the entropy field, respectively. The strains in the reference configuration $\mathscr{B}_0$ are measured by the right Cauchy-Green tensor $\mathbf{C}_t = \mathbf{F}_t^T \mathbf{F}_t$ based on the gradient $\mathbf{F}_t = \nabla \varphi_t$ of the deformation φ_t with respect to a point $\mathbf{X} \in \mathscr{B}_0$ in the reference configuration. To fulfill the Clausius-Plank inequality, the entropy field and the second Piola-Kirchhoff stress tensor field are given by

$$\eta_t = -\frac{\partial \Psi(\Theta_t, \mathbf{C}_t, \mathbf{\Gamma}_t)}{\partial \Theta_t} \qquad \mathbf{S}_t = 2\,\frac{\partial \Psi(\Theta_t, \mathbf{C}_t, \mathbf{\Gamma}_t)}{\partial \mathbf{C}_t} \tag{1}$$

respectively. The first Piola-Kirchhoff stress tensor $\mathbf{P}_t$ in the equations of motion then coincides with the transformation $\mathbf{F}_t \mathbf{S}_t$. Restricting to isotropic material, the free energy depends only on the three invariants of its tensor-valued arguments. We consider a free energy $\Psi(\Theta_t, I_1^{C_t}, I_2^{C_t}, I_3^{C_t}, I_1^{\Lambda_t}, I_2^{\Lambda_t}, I_3^{\Lambda_t})$ divided into the free energy in the thermodynamic equilibrium and non-equilibrium free energies. To model the memory effect, we apply the concept of a material isomorphism. In the case of isotropy, the internal variable then coincides with a symmetric tensor $\mathbf{\Gamma}_t$ and the free energy depends on the invariants of the tensor $\mathbf{\Lambda}_t = \mathbf{C}_t \mathbf{\Gamma}_t^{-1}$. The entropy flux corresponding to Fourier's law of isotropic heat conduction reads

$$\mathbf{H}_t = -\frac{1}{\Theta_t}\,\mathbf{K}_t \nabla \Theta_t = -\frac{1}{\Theta_t}\,k_0 \sqrt{I_3^{C_t}}\,\mathbf{C}_t^{-1} \nabla \Theta_t \tag{2}$$

which leads in the Clausius-Plank inequality to a quadratic dissipation $D_t^{\text{con}} = -\nabla \Theta_t\, \mathbf{H}_t$ arising from conduction of heat. To satisfy the Clausius-Plank inequality, we introduce the internal dissipation D_t^{int} associated with the memory effect as

$$D_t^{\text{int}} = \left[\frac{1}{2}\,\dot{\mathbf{\Gamma}}_t\,\mathbf{\Gamma}_t^{-1}\right] : \mathbb{V} : \left[\frac{1}{2}\,\dot{\mathbf{\Gamma}}_t\,\mathbf{\Gamma}_t^{-1}\right] \doteq \left[\frac{1}{2}\,\dot{\mathbf{\Gamma}}_t\,\mathbf{\Gamma}_t^{-1}\right] : \mathbf{M}_t \tag{3}$$

The viscosity tensor $\mathbb{V}$ as well as the conductivity tensor $\mathbf{K}_t$ of the thermo-viscoelastic material are thus introduced as positive-definite symmetric bilinear forms. According to the Clausius-Plank inequality the stress tensor $\mathbf{M}_t$ in Equation[3] has to fulfill the identity

$$\mathbf{M}_t = -2\,\frac{\partial \Psi(\Theta_t, \mathbf{C}_t, \mathbf{\Gamma}_t)}{\partial \mathbf{\Gamma}_t}\,\mathbf{\Gamma}_t \tag{4}$$

The enforcement of this identity is the task of the viscous evolution equation.

3 SPACE-TIME WEAK EQUATIONS

The balance equations of continuum dynamics furnish the strong forms of the equation of motion and of the entropy evolution equation as three PDE first-order in time. The fourth differential equation in time is obtained from the mentioned condition on the stress tensor $\mathbf{M}_t$ as ODE at each considered point $\mathbf{X} \in \mathscr{B}_0$. This coupled system of differential equations fulfills the stability estimate

$$\mathcal{H}(\chi_{t=t_{n+1}}) - \mathcal{H}(\chi_{t=t_n}) = \int_{\mathscr{T}_n} \int_{\mathscr{B}_0} \frac{\Theta_\infty}{\Theta_t}\,(D_t^{\text{con}} + D_t^{\text{int}}) \leqslant 0 \tag{5}$$

where $\mathscr{T}_n = [t_n, t_{n+1}]$ denotes any subinterval in the time interval $\mathscr{T} = [t_0, T]$ of interest. The state field $\chi_t = (\varphi_t, \pi_t, \eta_t, \mathbf{\Gamma}_t)$ include the momentum field $\pi_t = \rho_0 \mathbf{v}_t$ instead of the corresponding velocity field $\mathbf{v}_t$. The scalar-valued field ρ_0 designates the density field in $\mathscr{B}_0$. The corresponding Lyapunov function $\mathcal{H}$ is the total energy

$$\mathcal{H}(\chi_t) = \int_{\mathscr{B}_0} \pi_t \cdot \mathbf{v}_t - \frac{1}{2}\,\rho_0 \mathbf{v}_t \cdot \mathbf{v}_t + e(\eta_t, \mathbf{C}_t(\nabla\varphi_t), \mathbf{\Gamma}_t) \tag{6}$$

We derive the space-time weak forms by employing the strong forms of the evolution equations in Equation[5]. We arrive at fully weak equations of motion, given by

$$\boxed{\int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \delta\dot{\pi}_t \cdot \dot{\varphi}_t = \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \delta\dot{\pi}_t \cdot \mathbf{v}_t} \qquad \boxed{\int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \dot{\pi}_t \cdot \delta\dot{\varphi}_t = -\int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \mathbf{P}_t : \nabla[\delta\dot{\varphi}_t]} \tag{7}$$

The test functions $\delta\dot{\varphi}_t$ and $\delta\dot{\pi}_t$ has to be time-dependent variations of $\dot{\varphi}_t$ and $\dot{\pi}_t$, respectively. Hence we have to use a cG method (compare Reference[1]). Since D_t^{con} is quadratic in $\nabla\vartheta_t$, the test function $\delta\vartheta_t$ of the entropy evolution equation is a time-dependent variation of the field $\vartheta_t = \Theta_t - \Theta_\infty$. Hence we obtain a dG approximation of η_t and Θ_t, respectively. However to fulfill Equation[5], the jump at $t = t_n$ of the dG method has to be formulated in the energy e. Hence we get

$$\boxed{\int_{\mathscr{B}_0} \frac{\delta\vartheta_{t=t_n^+}}{\vartheta_{t=t_n^+}}\, [\![e]\!]_{t=t_n^+} + \int_{\mathscr{T}_n^+}\int_{\mathscr{B}_0} \dot{\eta}_t\,\delta\vartheta_t = \int_{\mathscr{T}_n^+}\int_{\mathscr{B}_0} \nabla[\delta\vartheta_t]\cdot\frac{\Theta_\infty}{\Theta_t}\,\mathbf{H}_t + \delta\vartheta_t\,\frac{D_t^{\text{int}}}{\Theta_t}} \tag{8}$$

where $\mathscr{T}_n^+ = [t_n^+, t_{n+1}]$ denotes the time interval behind the jump. As the internal Dissipation D_t^{int} is quadratic in the time derivative $\dot{\mathbf{\Gamma}}_t$ we determine with the cG approximation

$$\boxed{\int_{\mathscr{T}_n} \left[\frac{1}{2}\,\delta\mathbf{\Gamma}_t\,\mathbf{\Gamma}_t^{-1}\right] : \mathbb{V} : \left[\frac{1}{2}\,\dot{\mathbf{\Gamma}}_t\,\mathbf{\Gamma}_t^{-1}\right] = \int_{\mathscr{T}_n} \left[\frac{1}{2}\,\delta\mathbf{\Gamma}_t\,\mathbf{\Gamma}_t^{-1}\right] : \mathbf{M}_t} \tag{9}$$

the internal variable evolution at each considered point $\mathbf{X} \in \mathscr{B}_0$ in an energy consistent way.

4 SPACE-TIME APPROXIMATION

We approximate the unknown fields and the constitutive laws such that Equation[5] is fulfilled in spite of numerical quadrature. We approximate the unknown fields φ_t, $\mathbf{v}_t$ and Θ_t by the spatial Lagrangian shape functions N_A. For the temporal approximation we use the Lagrangian shape functions M_I as for the approximation of $\mathbf{\Gamma}_t$, η_t and $\mathbf{C}_t$. The test functions $\delta\dot{\varphi}_t$, $\delta\dot{\pi}_t$ and $\delta\,\mathbf{\Gamma}_t$ are approximated in time by reduced Lagrangian shape functions $\tilde{M}_J$, however $\delta\vartheta_t$ is approximated by the functions M_I. We derive special approximations $\hat{\mathbf{S}}_t$ and $\hat{D}_t^{\text{int}}$ in Equation[7] and Equation[8], respectively, given by

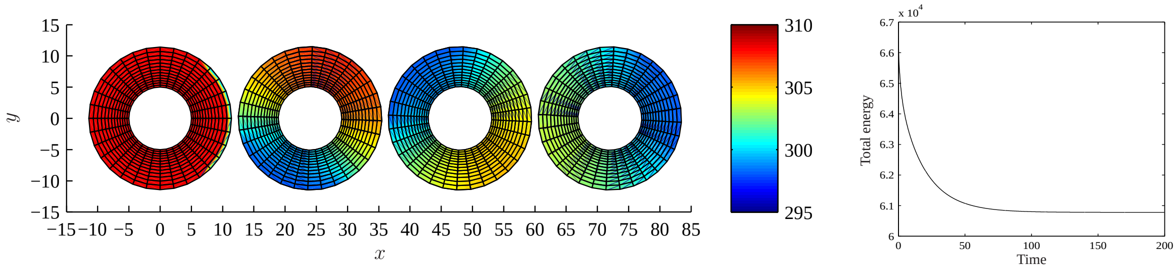
$$\hat{\mathbf{S}}_t = \mathbf{S}_t + 2\,\frac{e_{t=t_{n+1}} - e_{t=t_n^+} - \displaystyle\int_{\mathscr{T}_n} \mathbf{S}_t : \frac{1}{2}\,\dot{\overline{\mathbf{F}_t^T\mathbf{F}_t}} + \vartheta_t\,\dot{\eta}_t - \mathbf{M}_t : \left[\frac{1}{2}\,\dot{\boldsymbol{\Gamma}}_t\,\boldsymbol{\Gamma}_t^{-1}\right]}{\displaystyle\int_{\mathscr{T}_n} \dot{\mathbf{C}}_t : \dot{\overline{\mathbf{F}_t^T\mathbf{F}_t}}}\,\dot{\mathbf{C}}_t$$

$$\hat{D}_t^{\text{int}} = D_t^{\text{int}} + \frac{\displaystyle\int_{\mathscr{T}_n} D_t^{\text{int}} - \int_{\mathscr{T}_n} D_t^{\text{int}}}{\displaystyle\int_{\mathscr{T}_n} \left[\frac{\vartheta_t}{\Theta_t}\,\dot{\boldsymbol{\Gamma}}_t\,\boldsymbol{\Gamma}_t^{-1} : \dot{\boldsymbol{\Gamma}}_t\,\boldsymbol{\Gamma}_t^{-1}\right]^2}\,\frac{\vartheta_t}{\Theta_t}\left[\dot{\boldsymbol{\Gamma}}_t\,\boldsymbol{\Gamma}_t^{-1} : \dot{\boldsymbol{\Gamma}}_t\,\boldsymbol{\Gamma}_t^{-1}\right]^2 \tag{10}$$

The first is necessary because we apply numerically quadrature in general and the last because we need different quadrature rules in the dG and cG method.

5 NUMERICAL EXAMPLE

The simulation shows a rotating stiff tyre with a small temperature Dirichlet boundary. We initiate the motion by initial velocities. The colours indicate the absolute body temperature.



As Equation[5] predicts, the total energy is steady decreasing till the equilibrium state is reached.

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