AN ENERGY CONSISTENT HYBRID SPACE-TIME FINITE ELEMENT METHOD FOR NONLINEAR THERMO-VISCOELASTODYNAMICS

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Abstract. We present a new space-time fe discretisation of a finite thermo-viscoelastic coupled sytem. The discretisation fulfills a nonlinear stability estimate with respect to a Lyapunov function called the total energy. The coupled sytem of differential equations first-order in time consists of the equations of motion, the entropy evolution equation and an evolution equation for a viscous internal variable. A new hyprid Galerkin (hG) approximation composed of a continuous Galerkin (cG) and a new discontinuous Galerkin (dG) met hod supplemented by particular time approximations in the constitutive laws leads to long-time stable numerical schemes.

1 INTRODUCTION

For performing a time discretisation an advantage of the fe method in time is that higher-order accurate time approximations can be obtained in a natural way. A further advantage is that a cG method in time is inherently energy-momentum consistent for the equations of motion (see Reference1). Hence the cG method is a natural starting point to construct higher-order and energy-momentum-conserving time integrators (see Reference2). These time integrators, derived from the enhanced Galerkin (eG) method, turned out to be well suited for computing long-time runs in nonlinear elastodynamics. An energy consistent time approximation is also of great advantage for simulating dissipative material (see References3,4). We show that an energy consistent discretisation by using a new enhanced hybrid Galerkin (ehG) method is also advantageous for nonlinear dynamical systems with dissipation arising from conduction of heat and from a memory effect of viscous material.

2 CONSTITUTIVE EQUATIONS

In a Lagrangian setting, we consider a thermo-viscoelastic material described by Fourier’s law of heat conduction together with an internal variable formulation. We start with the relative internal energy \( e(\eta_t, C_t, \Gamma_t) = [\Theta_t - \Theta_\infty] \eta_t + \Psi(\Theta_t, C_t, \Gamma_t) \) emanating from a Legendre transform of the free energy \( \Psi(\Theta_t, C_t, \Gamma_t) \) with respect to the temperature \( \Theta_t \). The fields \( \Theta_\infty \)
Michael Groß and Peter Betsch

and $\eta_t$ denote the constant environment temperature field and the entropy field, respectively. The strains in the reference configuration $B_0$ are measured by the right Cauchy-Green tensor $C_t = F_t^T F_t$ based on the gradient $F_t = \nabla \varphi_t$ of the deformation $\varphi_t$ with respect to a point $X \in B_0$ in the reference configuration. To fulfill the Clausius-Plank inequality, the entropy field and the second Piola-Kirchhoff stress tensor field are given by

$$
\eta_t = -\frac{\partial \Psi(\Theta_t, C_t, \Gamma_t)}{\partial \Theta_t} \quad S_t = 2 \frac{\partial \Psi(\Theta_t, C_t, \Gamma_t)}{\partial C_t}
$$

respectively. The first Piola-Kirchhoff stress tensor $P_t$ in the equations of motion then coincides with the transformation $F_t S_t$. Restricting to isotropic material, the free energy depends only on the three invariants of its tensor-valued arguments. We consider a free energy $\Psi(\Theta_t, I_{C_t}^1, I_{C_t}^2, I_{C_t}^3, I_{\Lambda_t}^1, I_{\Lambda_t}^2, I_{\Lambda_t}^3)$ divided into the free energy in the thermodynamic equilibrium and non-equilibrium free energies. To model the memory effect, we apply the concept of a material isomorphism. In the case of isotropy, the internal variable then coincides with a symmetric tensor $\Gamma_t$ and the free energy depends on the invariants of the tensor $\Lambda_t = C_t \Gamma_t^{-1}$. The entropy flux corresponding to Fourier’s law of isotropic heat conduction reads

$$
H_t = -\frac{1}{\Theta_t} k_0 \sqrt{I_{C_t}^3} C_t^{-1} \nabla \Theta_t
$$

which leads in the Clausius-Plank inequality to a quadratic dissipation $D_t^{\text{con}} = -\nabla \Theta_t H_t$ arising from conduction of heat. To satisfy the Clausius-Plank inequality, we introduce the internal dissipation $D_t^{\text{int}}$ associated with the memory effect as

$$
D_t^{\text{int}} = \frac{1}{2} \dot{\Gamma}_t \Gamma_t^{-1} : \Gamma_t = \frac{1}{2} \dot{\Gamma}_t \Gamma_t^{-1} \cdot \Gamma_t
$$

The viscosity tensor $\nu$ as well as the conductivity tensor $K_t$ of the thermo-viscoelastic material are thus introduced as positive-definite symmetric bilinear forms. According to the Clausius-Plank inequality the stress tensor $M_t$ in Equation $3$ has to fulfill the identity

$$
M_t = -2 \frac{\partial \Psi(\Theta_t, C_t, \Gamma_t)}{\partial \Gamma_t} \Gamma_t
$$

The enforcement of this identity is the task of the viscous evolution equation.

3 SPACE-TIME WEAK EQUATIONS

The balance equations of continuum dynamics furnish the strong forms of the equation of motion and of the entropy evolution equation as three PDE first-order in time. The fourth differential equation in time is obtained from the mentioned condition on the stress tensor $M_t$ as ODE at each considered point $X \in B_0$. This coupled system of differential equations fulfills the stability estimate

$$
\mathcal{H}(\chi_{t=t_{n+1}}) - \mathcal{H}(\chi_{t=t_n}) = \int_{T_{t_n}} \int_{B_0} \frac{\partial \Psi}{\partial \Theta_t} (D_t^{\text{con}} + D_t^{\text{int}}) \leq 0
$$
where \( \mathcal{T}_n = [t_n, t_{n+1}] \) denotes any subinterval in the time interval \( \mathcal{T} = [t_0, T] \) of interest. The state field \( \chi_t = (\varphi_t, \pi_t, \eta_t, \Gamma_t) \) include the momentum field \( \pi_t = \rho_0 v_t \) instead of the corresponding velocity field \( v_t \). The scalar-valued field \( \rho_0 \) designates the density field in \( \mathcal{B}_0 \). The corresponding Lyapunov function \( \mathcal{H} \) is the total energy

\[
\mathcal{H}(\chi_t) = \int_{\mathcal{B}_0} \pi_t \cdot v_t - \frac{1}{2} \rho_0 v_t \cdot v_t + c(\eta_t, C_t(\nabla \varphi_t), \Gamma_t)
\]

We derive the space-time weak forms by employing the strong forms of the evolution equations in Equation\(^5\). We arrive at fully weak equations of motion, given by

\[
\begin{align*}
\int_{\mathcal{T}_n} \int_{\mathcal{B}_0} \delta \pi_t \cdot \dot{v}_t &= \int_{\mathcal{T}_n} \int_{\mathcal{B}_0} \delta \pi_t \cdot v_t \\
\int_{\mathcal{T}_n} \int_{\mathcal{B}_0} \dot{\pi}_t \cdot \delta v_t &= - \int_{\mathcal{T}_n} \int_{\mathcal{B}_0} \mathbf{P}_t : \nabla [\delta \dot{\varphi}_t]
\end{align*}
\]

The test functions \( \delta \dot{v}_t \) and \( \delta \pi_t \) has to be time-dependent variations of \( \dot{v}_t \) and \( \pi_t \), respectively. Hence we use a cG method (compare Reference\(^1\)). Since \( D_t^{\text{int}} \) is quadratic in \( \nabla \dot{\varphi}_t \), the test function \( \delta \dot{\varphi}_t \) of the entropy evolution equation is a time-dependent variation of the field \( \dot{\varphi}_t = \Theta_t - \Theta_\infty \). Hence we obtain a dG approximation of \( \eta_t \) and \( \Theta_t \), respectively. However to fulfill Equation\(^5\), the jump at \( t = t_n \) of the dG method has to be formulated in the energy \( e \). Hence we get

\[
\int_{\mathcal{T}_n} \int_{\mathcal{B}_0} \delta \eta_t \cdot \dot{\varphi}_t = \int_{\mathcal{T}_n} \int_{\mathcal{B}_0} \delta \eta_t \cdot \pi_t = \int_{\mathcal{T}_n} \int_{\mathcal{B}_0} \nabla [\delta \dot{\varphi}_t] \cdot \frac{\Theta_\infty}{\Theta_t} H_t + \delta \dot{\varphi}_t \frac{D_t^{\text{int}}}{\Theta_t}
\]

where \( \mathcal{T}_n^+ = [t_n^+, t_{n+1}] \) denotes the time interval behind the jump. As the internal Dissipation \( D_t^{\text{int}} \) is quadratic in the time derivative \( \Gamma_t \) we determine with the cG approximation

\[
\int_{\mathcal{T}_n} \left[ \frac{1}{2} \delta \Gamma_t \Gamma_t^{-1} \right] : \nabla \left[ \frac{1}{2} \Gamma_t \Gamma_t^{-1} \right] = \int_{\mathcal{T}_n} \left[ \frac{1}{2} \delta \Gamma_t \Gamma_t^{-1} \right] : M_t
\]

the internal variable evolution at each considered point \( X \in \mathcal{B}_0 \) in an energy consistent way.

4 SPACE-TIME APPROXIMATION

We approximate the unknown fields and the constitutive laws such that Equation\(^5\) is fulfilled in spite of numerical quadrature. We approximate the unknown fields \( \varphi_t, v_t \) and \( \Theta_t \) by the spatial Lagrangian shape functions \( N_A \). For the temporal approximation we use the Lagrangian shape functions \( M_I \) as for the approximation of \( \Gamma_t, \eta_t \) and \( C_t \). The test functions \( \delta \dot{\varphi}_t, \delta \pi_t \) and \( \delta \Gamma_t \) are approximated in time by reduced Lagrangian shape functions \( \bar{M}_{\dot{\varphi}} \), however \( \delta \dot{\varphi}_t \) is approximated by the functions \( M_I \). We derive special approximations \( \bar{S}_t \) and \( \bar{D}_{t}^{\text{int}} \) in Equation\(^7\)
and Equation\(^8\), respectively, given by

\[
\hat{S}_t = S_t + 2 \left( \int_{\mathcal{T}_t} \dot{S}_t : \frac{1}{2} \mathbf{F}_t^T \mathbf{F}_t \, dt + \partial_t \dot{\mathbf{u}}_t - \mathbf{M}_t : \left[ \frac{1}{2} \hat{\mathbf{G}}_t \mathbf{G}_t^{-1} \right] \mathbf{C}_t \right)
\]

\[
\hat{D}^{int}_t = D^{int}_t + \left( \int_{\mathcal{T}_t} \hat{D}^{int}_t - \int_{\mathcal{T}_t} D^{int}_t \, dt \right) \frac{\theta_t}{\theta_t} \left[ \hat{\mathbf{G}}_t \mathbf{G}_t^{-1} : \hat{\mathbf{G}}_t \mathbf{G}_t^{-1} \right]^2
\]

Equation (10)

The first is necessary because we apply numerically quadrature in general and the last because we need different quadrature rules in the dG and cG method.

5 NUMERICAL EXAMPLE

The simulation shows a rotating stiff tyre with a small temperature Dirichlet boundary. We initiate the motion by initial velocities. The colours indicate the absolute body temperature.

As Equation\(^5\) predicts, the total energy is steady decreasing till the equilibrium state is reached.

REFERENCES


